Answer on Question #66338 – Math – Differential Equations

Question

Solve the differential equation

1)

$$x^2 p + y^2 q = (x + y)z$$

Solution

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x + y)z$$

Equation of characteristics:

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x + y)z}$$

Then:

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

$$-\frac{1}{x} = -\frac{1}{y} + C_1$$

$$C_1 = \frac{x - y}{xy}$$

$$y = \frac{x}{C_1 x + 1}$$

$$\frac{dx}{x^2} = \frac{dz}{(x + y)z} = \frac{dz}{\left(x + \frac{x}{C_1 x + 1}\right) z}$$

$$\frac{C_1 x^2 + 2x}{x^2 (C_1 x + 1)} dx = \frac{C_1 x + 2}{x(C_1 x + 1)} = \frac{dz}{z}$$

$$\int \frac{C_1 x + 2}{x(C_1 x + 1)} dx = \int \frac{dz}{z}$$

$$\frac{C_1x+2}{x(C_1x+1)}=\frac{1}{x}+\frac{1}{x(C_1x+1)}$$

$$\frac{1}{x(C_1x+1)}=\frac{A}{x}+\frac{B}{C_1x+1}$$

$$A(C_1x+1)+Bx=1$$

$$A=1\ ;\ \ B=-C_1$$

$$\frac{1}{x(C_1x+1)}=\frac{1}{x}-\frac{C_1}{C_1x+1}$$

$$\int\left(\frac{2}{x}-\frac{C_1}{C_1x+1}\right)dx=\int\frac{dz}{z}$$

$$2\ln|x|-\ln|C_1x+1|=\ln|z|+\ln C_2$$

$$C_2=\frac{x^2}{z(C_1x+1)}=\frac{x^2}{z\left(\frac{x-y}{xy}x+1\right)}=\frac{yx^2}{zx}=\frac{xy}{z}$$

Answer: $F\left(\frac{x-y}{xy},\frac{xy}{z}\right)=0$

Question

2)

$$p^{1/2}-q^{1/2}+3x=0$$

Solution

$$\left(\frac{\partial z}{\partial x}\right)^{1/2}-\left(\frac{\partial z}{\partial y}\right)^{1/2}+3x=0$$

Equations of characteristics:

$$\dot{x}=\frac{dx}{d\tau}=\frac{1}{2\sqrt{p}}\ ;\ \dot{y}=\frac{dy}{d\tau}=-\frac{1}{2\sqrt{q}}\ ;\ \dot{p}=\frac{dp}{d\tau}=-3\ ;\ \dot{q}=\frac{dq}{d\tau}=0\ ;\ \dot{z}=\frac{dz}{d\tau}=\frac{\sqrt{p}-\sqrt{q}}{2}$$

Then:

$$p = -3\int d\tau = -3\tau + C_1$$

$$q = C_2$$

$$y = -\int \frac{d\tau}{2\sqrt{q}} = -\frac{\tau}{2\sqrt{q}} + C_3 = -\frac{\tau}{2\sqrt{C_2}} + C_3$$

$$x = \int \frac{d\tau}{2\sqrt{p}} = \int \frac{d\tau}{2\sqrt{3\tau + C_1}} = -\frac{1}{6}\sqrt{3\tau + C_1} + C_4$$

$$z = \int \frac{\sqrt{p} - \sqrt{q}}{2} d\tau = \int \frac{\sqrt{3\tau + C_1} - \sqrt{C_2}}{2} d\tau = -\frac{1}{9}(-3\tau + C_1)^{3/2} - \frac{1}{2}\tau\sqrt{C_2} + C_5$$

$$\sqrt{3\tau + C_1} = 6(C_4 - x) \quad ; \quad \tau = 2\sqrt{C_2}(C_3 - y)$$

$$z = -24(C_4 - x)^3 - C_2(C_3 - y) + C_5$$

Answer: $z = 24(x - c_1)^3 + c_2 y + c_3; \quad c_2 = q.$

Reference:

E. A. Kuznetsov, D. A. Shapiro, Methods of Mathematical Physics, Part I, Chapter 2, 2011.