Answer on Question #66333 – Math – Differential Equations

Question

Verify that the pfaffian differential equation yz dx+ (x^2y-zx) dy +(x^2z-xy) dz = 0 is integrable and hence find its integral.

Solution

We have

$$yzdx + (x^2y - zx)dy + (x^2z - xy)dz = 0$$

General form of the Pfafffian equation is

$$Pdx + Qdy + Rdz = 0$$

The integrability condition for the Pfaffian equation is [1, page 384]

$$(\mathrm{curl}F, F) = 0$$

where $F = (P, Q, R)$, or

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

Verify this condition for the given equation. We get

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) =$$

$$= yz(-x + x) + (x^2y - zx)(2xz - y - y) + (x^2z - xy)(z - 2xy + z) =$$

$$= 2x(xy - z)(xz - y) + x(xz - y)2(z - xy) = 0$$

The integrability condition for this equation hold.

If the Pfaffian equation is multiplied by a certain function $\mu(x, y, z)$ then one can obtain in the left-hand side the total differential

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = du = 0$$

That gives the solution of the Pfaffian equation $u = const$

multiply the original equation by $\frac{1}{x^2}$. We get

$$\frac{yz}{x^2}dx + \left(y - \frac{z}{x}\right)dy + \left(z - \frac{y}{x}\right)dz = 0$$

$$\left(\frac{yz}{x^2}dx - \frac{z}{x}dy - \frac{y}{x}dz\right) + ydy + zdz$$

$$d\left(-\frac{yz}{x}\right) + d\left(\frac{y^2}{2}\right) + d\left(\frac{z^2}{2}\right) = 0$$

$$d\left(-\frac{yz}{x} + \frac{y^2}{2} + \frac{z^2}{2}\right) = 0$$

Finally we get solution

$$-\frac{yz}{x} + \frac{y^2}{2} + \frac{z^2}{2} = C$$

Answer: The differential equation is integrable. The integral of the original equation is

$$-\frac{yz}{x} + \frac{y^2}{2} + \frac{z^2}{2} = C$$

Reference:

[1] Daniel Zwillinger. Handbook of Differential Equations, 3rd edition