Answer on Question #66191 – Math – Calculus

Question

3 (a) Obtain the largest possible domain and range of the function f, defined by $f(x)=\sqrt{\frac{x+1}{x+2}}$.
Further, check whether or not $\lim\limits_{x\to a} f(x)$ exists for $a=-1,-2$.

Solution

The function is well defined for all x such that

$$\frac{x+1}{x+2}\geq 0 \text{ and } x\neq -2.$$

Solving the first inequality yields

$$\frac{x+1}{x+2}\geq 0 \leftrightarrow \begin{cases}(x+1)(x+2)\geq 0,\\ x\neq -2\end{cases} \leftrightarrow x\,\epsilon\,(-\infty,-2)\cup[-1,+\infty)$$

Hence $D(f)=(-\infty,-2)\cup[-1,+\infty)$. We also have $\lim\limits_{x\to -1}\sqrt{\frac{x+1}{x+2}}=0$ and $\lim\limits_{x\to -2}\sqrt{\frac{x+1}{x+2}}=\infty$.

In order to find the range of f we solve the equation

$$\sqrt{\frac{x+1}{x+2}}=a$$

with respect to x for any nonnegative a. Then

$$\frac{x+1}{x+2}=a^2 \leftrightarrow x+1=a^2(x+2) \leftrightarrow (1-a^2)x=2a^2-1 \leftrightarrow x=\frac{2a^2-1}{1-a^2}.$$

Therefore the equation has no solution for $a=1$. Any other nonnegative value can be reached by f. Hence

$R(f)=[0,1)\cup(1,+\infty)$.

Answer: $D(f)=(-\infty,-2)\cup[-1,+\infty)$, $R(f)=[0,1)\cup(1,+\infty)$, $\lim\limits_{x\to -1}\sqrt{\frac{x+1}{x+2}}=0$,
$\lim\limits_{x\to -2}\sqrt{\frac{x+1}{x+2}}=\infty$.

Question

3(b) Find

$$\frac{df}{dx},$$

where $f(x)=\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$.

Solution

First we rewrite the function in the form

$$f(x)=\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)=\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)=\cos^{-1}\left(1-\frac{2}{x^2+1}\right).$$

By the chain rule we have

$$f'(x) = -\frac{1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}} \cdot \left(1-\frac{2}{x^2+1}\right)' = -\frac{x^2+1}{\sqrt{(x^2+1)^2-(x^2-1)^2}} \cdot \frac{4x}{(x^2+1)^2} =$$

$$= \frac{-4x}{\sqrt{4x^2}(x^2+1)} = \frac{-2x}{|x| \cdot (x^2+1)} = \begin{cases} \frac{2}{x^2+1} & if \ \ x < 0, \\ -\frac{2}{x^2+1} & if \ x > 0. \end{cases}$$

Note that the function is not defined at $x = 0$.

Answer: $\frac{df}{dx}(x) = \begin{cases} \frac{2}{x^2+1} & if \ \ x < 0, \\ -\frac{2}{x^2+1} & if \ x > 0. \end{cases}$