Answer on Question #66150 – Math – Differential Equations

Question

Solve the equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x^m$$

for all positive integer values of m.

Solution

We have the differential equation (DE)

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x^m$$

This is the Euler equation. Change the variable x by t:

$$x=e^t,\ \ t=\ln x,\ \ (x>0)$$

Express the derivatives with respect to x in terms of the derivatives with respect to t:

$$\frac{d}{dx}=\frac{d}{dt}\frac{dt}{dx}=\frac{d\ln x}{dx}\frac{d}{dt}=\frac{1}{x}\frac{d}{dt}=e^{-t}\frac{d}{dt}$$

$$\frac{d^2}{dx^2}=\frac{d}{dx}\left(\frac{d}{dx}\right)=e^{-t}\frac{d}{dt}\left(e^{-t}\frac{d}{dt}\right)=e^{-2t}\frac{d^2}{dt^2}-e^{-2t}\frac{d}{dt}$$

Substituting in the equation we get

$$e^{2t}\left(e^{-2t}\frac{d^2y}{dt^2}-e^{-2t}\frac{dy}{dt}\right)+e^te^{-t}\frac{dy}{dt}+y=e^{mt}$$

$$\frac{d^2y}{dt^2}+y=e^{mt}\ (1)$$

To solve a nonhomogeneous linear differential equation (1) we must do the following steps:
1) find the complementary function y_c that is the general solution of the associated homogeneous DE

$$\frac{d^2y}{dt^2}+y=0$$

2) find any particular solution y_p of the nonhomogeneous equation

$$\frac{d^2y}{dt^2}+y=e^{mt}$$

The solution of the associated homogeneous DE is

$$y_c=C_1\cos t+C_2\sin t,$$

where C_1 and C_2 are arbitrary real constants.

Assume for the particular solution

$$y_p=Ae^{mt}$$

Substituting y_p into the given differential equation (1), we get

$$\frac{d^2y}{dt^2}+y=Am^2e^{mt}+Ae^{mt}=e^{mt},$$

that is,

$$Am^2+A=1,$$

hence

$$A = \frac{1}{m^2+1}.$$

Then

$$y_p = \frac{1}{m^2+1} e^{mt}.$$

The general solution of the nonhomogeneous equation is

$$y = C_1 \text{cos}t + C_2 \text{sin}t + \frac{1}{m^2+1} e^{mt}$$

Substituting $t = \ln x$ we get

$$y = C_1 \cos(\ln x) + C_2 \sin(\ln x) + \frac{1}{m^2+1} x^m$$

Answer:

$$y = C_1 \cos(\ln x) + C_2 \sin(\ln x) + \frac{1}{m^2+1} x^m$$