Answer on Question 66149 - Math - Calculus

Question: Find the integrating factor of the differential equation

$$(6xy - 3y^2 + 2y)dx + 2(x - y)dy = 0$$

and hence solve it.

Solution: Set $P(x,y) = 6xy - 3y^2 + 2y$ and $Q(x,y) = 2(x-y)$. The equation

$$(6xy - 3y^2 + 2y)dx + 2(x - y)dy = 0$$

is not exact, because

$$\frac{\partial}{\partial y}P(x,y) \neq \frac{\partial}{\partial x}Q(x,y).$$

Indeed,

$$\frac{\partial P}{\partial y}(x,y) = \frac{\partial}{\partial y}(6xy - 3y^2 + 2y) = 6x - 6y + 2,$$

$$\frac{\partial Q}{\partial x}(x,y) = \frac{\partial}{\partial x}(2x - 2y) = 2.$$

Nevertheless, the equation can be turned into in an exact equation if it is multiplied by the correct integrating factor.

Suppose that there exists an integrating factor I that depends only on x. Then I is a solution of the equation

$$\frac{dI}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)I.$$

This equation will have a solution that depends only on x provided that

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

depends only on x. For our equation we have

$$\frac{1}{Q(x,y)}\left(\frac{\partial P}{\partial y}(x,y) - \frac{\partial Q}{\partial x}(x,y)\right) = \frac{6x - 6y + 2 - 2}{2(x-y)} = \frac{6(x-y)}{2(x-y)} = 3.$$

Hence, I is a solution of the equation $I' = 3I$, and $I(x) = e^{3x}$. Multiplying the equation by e^{3x} yields

$$(6xy - 3y^2 + 2y)e^{3x}\,dx + 2(x - y)e^{3x}\,dy = 0,$$

which is now exact, since

$$\frac{\partial}{\partial y}\{(6xy - 3y^2 + 2y)e^{3x}\} = (6x - 6y + 2)e^{3x} = \frac{\partial}{\partial x}\{2(x-y)e^{3x}\}.$$

To find the potential $U(x, y)$, we first integrate

$$\frac{\partial U}{\partial x} = (6xy - 3y^2 + 2y)e^{3x}$$

partially with respect to x to give

$$U(x, y) = (2xy - y^2)e^{3x} + \alpha(y).$$

Indeed,

$$\int (6xy - 3y^2 + 2y)e^{3x}\,dx = 6y\int xe^{3x}\,dx - (3y^2 - 2y)\int e^{3x}\,dx$$

$$= 2y\int x\,de^{3x} - \frac{3y^2 - 2y}{3}e^{3x} = 2xye^{3x} - 2y\int e^{3x}\,dx - y^2e^{3x} + \frac{2}{3}ye^{3x} + \alpha(y)$$

$$= 2xye^{3x} - \frac{2}{3}ye^{3x} - y^2e^{3x} + \frac{2}{3}ye^{3x} + \alpha(y) = (2xy - y^2)e^{3x} + \alpha(y).$$

To fix $\alpha(y)$ we differentiate U partially with respect to y, and substitute the result into equality $\frac{\partial U}{\partial y} = Q$,

$$\frac{\partial U}{\partial y} = (2x - 2y)e^{3x} + \alpha'(y) = 2(x - y)e^{3x}.$$

We therefore have $\alpha'(y) = 0$ and $\alpha(y) = C$. So we finally have our solution $y(2x - y)e^{3x} = C$.

Answer: The integrating factor is e^{3x}. The general solution in the implicit form is $y(2x - y)e^{3x} = C$.