Answer on Question #66148 – Math – Calculus

Question

Solve the following equation by changing the independent variable

$$(1+x^2)^2y" + 2x(1+x^2)y' + +4y = 0$$

Solution

Let

$$t = \arctan(x).$$

We need to express derivatives $\frac{\mathrm{dy}}{\mathrm{dx}}, \frac{\mathrm{d^2y}}{\mathrm{dx^2}}$ trough $\frac{\mathrm{d^2y}}{\mathrm{dt^2}}, \ \frac{\mathrm{dy}}{\mathrm{dt}}, \ \frac{\mathrm{d^2t}}{\mathrm{dx^2}}, \ \frac{\mathrm{dt}}{\mathrm{dx}}$:

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{dt}}\frac{\mathrm{dt}}{\mathrm{dx}} = \frac{dy}{dt}\frac{1}{1+x^2};$$

$$\frac{\mathrm{d^2y}}{\mathrm{dx^2}} = \frac{\mathrm{d^2y}}{\mathrm{dt^2}}\left(\frac{\mathrm{dt}}{\mathrm{dx}}\right)^2 + \frac{\mathrm{dy}}{\mathrm{dt}}\frac{\mathrm{d^2t}}{\mathrm{dx^2}} = \frac{\mathrm{d^2y}}{\mathrm{dt^2}}\frac{1}{(1+x^2)^2} - \frac{\mathrm{dy}}{\mathrm{dt}}\frac{2\mathrm{x}}{(1+\mathrm{x}^2)^2};$$

So, now we have

$$(1+x^2)^2\left(\frac{\mathrm{d^2y}}{\mathrm{dt^2}}\frac{1}{(1+x^2)^2} - \frac{\mathrm{dy}}{\mathrm{dt}}\frac{2\mathrm{x}}{(1+\mathrm{x}^2)^2}\right) + 2x(1+x^2)\frac{dy}{dt}\frac{1}{1+x^2} + 4y = 0;$$

$$\frac{\mathrm{d^2y}}{\mathrm{dt^2}} - 2x\frac{dy}{dt} + 2x\frac{dy}{dt} + 4y = 0;$$

$$\frac{\mathrm{d^2y}}{\mathrm{dt^2}} + 4y = 0.$$

The characteristic equation is

$$\lambda^2 + 4 = 0 \ \Rightarrow \lambda_1 = 2i, \lambda_2 = -2i;$$

The general solution is

$$y(t) = C_1\cos(2t) + C_2\sin(2t),$$

where C_1 and C_2 are arbitrary real constants.

In terms of x rewrite the solution in the following way:

$$y(x) = C_1\cos(2\arctan(x)) + C_2\sin(2\arctan(x)).$$

Answer: $y(x) = C_1\cos(2\arctan(x)) + C_2\sin(2\arctan(x))$.

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