Answer on Question #66141 – Math – Differential Equations

Question

Solve

$$\frac{dy}{dx} + xy = \frac{y^2 e^{x^2}}{2\sin x}$$

Solution

$\frac{dy}{dx} + xy = \frac{y^2 e^{x^2}}{2\sin x}$,

$\frac{y\prime}{y^2} + \frac{x}{y} = \frac{e^{x^2}}{2\sin x}$

is the first-order nonlinear ordinary differential equation

First we solve the linear differential equation in order to use the method of variation of the constant

$$\frac{y'}{y^2} + \frac{x}{y} = 0$$

$$\frac{y'}{y} + x = 0$$

$$\frac{dy}{y} + xdx = 0$$

$$\int \frac{dy}{y} + \int xdx = C$$

$$\ln(\mathrm{y}) = -\frac{\mathrm{x}^2}{2} + C$$

$y = Ae^{\frac{-x^2}{2}}$,

C and A are arbitrary constants of integration

$$y = Ae^{\frac{-x^2}{2}}$$

We use the method of variation of constant. Let A be a function which depends on x:

$$A = A(x)$$

We find the derivative and substitute y and y' in the original equation:

$$y' = A'e^{\frac{-x^2}{2}} + A(-x)e^{\frac{-x^2}{2}}$$

$$A'e^{\frac{-x^2}{2}} + A(-x)e^{\frac{-x^2}{2}} + Axe^{\frac{-x^2}{2}} = \frac{e^{x^2}}{2\sin x}\,(Ae^{\frac{-x^2}{2}})^2$$

$$A' e^{\frac{-x^2}{2}} = \frac{e^{x^2}}{2\sin x} A^2 e^{-x^2}$$

Solving the equation

$$\frac{dA}{A^2} = \frac{e^{\frac{x^2}{2}} dx}{2\sin x}$$

$$\frac{-1}{A} = \int \frac{e^{\frac{x^2}{2}} dx}{2\sin x} + B,$$

B is an integration constant.

The integral is not expressed in elementary functions. Therefore we introduce

$$I(x) = \int \frac{e^{\frac{x^2}{2}} dx}{2\sin x}.$$

Finally we get

$$\frac{-1}{A} = I(x) + B,$$

$$A = \frac{-1}{I(x)+B},$$

$$\frac{y(x)}{e^{-\frac{x^2}{2}}} = -\frac{1}{I(x)+B},$$

$$y(x) = \frac{-e^{\frac{-x^2}{2}}}{I(x)+B}.$$

Answer: $y(x) = \frac{-e^{\frac{-x^2}{2}}}{I(x)+B}$.