Answer on Question #66102–Math–Calculus
Question

Find the directional derivative of $f(x,y)=\frac{2xy}{x^2+y^2},(x,y)\neq(0,0)$ and $0,\ (x,y)=(0,0)$

at $(0,0)$ in the direction of

(i) θ=π/2;

(ii) **(ii)** θ= π/4 .

Solution

By the definition of partial derivatives we get

$$f'_x(0,0)=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to0}\frac{\frac{2x\cdot0}{x^2+0^2}-0}{x}=\lim_{x\to0}\frac{0}{x}=0;$$

$$f'_y(0,0)=\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to0}\frac{\frac{2y\cdot0}{y^2+0^2}-0}{y}=\lim_{y\to0}\frac{0}{y}=0;$$

Consider the increment of the function f at the point $(0,0)$:

$\Delta f(0,0)=\frac{2xy}{x^2+y^2}=a(x,y)\cdot r$, where $a(x,y)=\frac{2xy}{(x^2+y^2)^{\frac{3}{2}}};\ r=(x^2+y^2)^{\frac{1}{2}}$.

Note that $r(\frac{1}{n},\frac{1}{n})\to0\,(n\to\infty)$ but $a(\frac{1}{n},\frac{1}{n})=\frac{2\cdot\frac{1}{n}\cdot\frac{1}{n}}{(\frac{1}{n^2}+\frac{1}{n^2})^{\frac{3}{2}}}=\frac{2\cdot\frac{1}{n^2}}{(\frac{2}{n^2})^{\frac{3}{2}}}=\frac{n}{\sqrt{2}}$ does not tend to 0,

then $a(x,y)$ does not infinitesimal as $(x^2+y^2)^{\frac{1}{2}}\to0$, hence f does not differentiable at $(0,0)$.

Solve this problem by the definition of the directional derivative of f:

The directional derivative at a in the direction l is the number

$$\frac{\partial f}{\partial l}(a)=\lim_{t\to+0}\frac{f(a+tl)-f(a)}{t}.$$

(i) θ=π/2

This direction is given by the unit vector $l=(0,1)$, so we get

$$\frac{\partial f}{\partial l}(0,0)=\lim_{t\to+0}\frac{f((0,0)+t(0,1))-f(0,0)}{t}=\lim_{t\to+0}\frac{f(0,t)-f(0,0)}{t}=\lim_{t\to+0}\frac{\frac{2t\cdot0}{0^2+t^2}-0}{t}=0;$$

(ii) θ= π/4

This direction is given by the unit vector $l=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, so we get

$$\frac{\partial f}{\partial l}(0,0)=\lim_{t\to+0}\frac{f((0,0)+t(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}))-f(0,0)}{t}=\lim_{t\to+0}\frac{f(\frac{t}{\sqrt{2}},\frac{t}{\sqrt{2}})-f(0,0)}{t}=$$

$$=\lim_{t\to+0}\frac{\frac{2\frac{t}{\sqrt{2}}\cdot\frac{t}{\sqrt{2}}}{\frac{t^2}{2}+\frac{t^2}{2}}-0}{t}=\lim_{t\to+0}\frac{1}{t}=+\infty,$$

so the directional derivative does not exist in this case.

Answer: (i) 0; **(ii)** it does not exist.