Answer on Question #66095 – Math – Calculus

Question

Find the moment of inertia I2 for the solid above the xy-plane bounded by the paraboloid z=x^2+y^2 and the cylinder x^2+y^2 =9 assuming the mean density to be constant C.

Solution

The moment of inertia of the mass

$$\Delta m_k = \rho(x_k, y_k, z_k)\Delta V_k$$

where $\rho(x_k, y_k, z_k)$is the density of an object occupying a region D in space (mass per unit volume), above the xy -plane is approximately [1, page 1109]

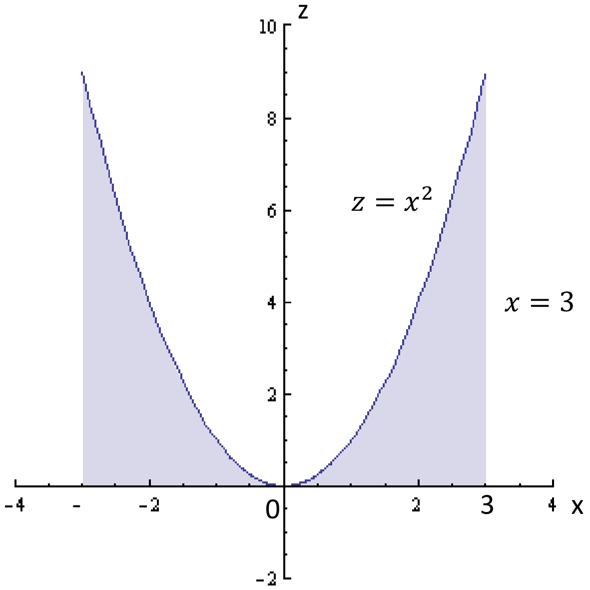
$$\Delta I_k = z^2(x_k, y_k, z_k)\Delta m_k = z^2(x_k, y_k, z_k)\rho(x_k, y_k, z_k)\Delta V_k$$

where $z(x_k, y_k, z_k)$ is the distance from the point (x_k, y_k, z_k) in D to a xy -plane.

The moment of inertia above the xy -plane of the entire object is [1, page 1109; 2]

$$I = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta I_k = \lim_{n\to\infty} \sum_{k=1}^{n} z^2(x_k, y_k, z_k)\rho(x_k, y_k, z_k)\Delta V_k = \iiint_D z^2\rho dV = C\iiint_D z^2 dV$$

since $\rho = C$. The region of integration D is bounded by surfaces $z = x^2 + y^2$ and $x^2 + y^2 = 9$. The cross section of this solid is shown in the figure.



Write this integral using cylindrical coordinates. The limits of integration with respect to z are $z = 0$ and $z = r^2$. The limits of integration with respect to r are 0 and 3

$$I = C\int_0^{2\pi}\left(\int_0^3\left(\int_0^{r^2} z^2 dz\right) r dr\right) d\theta = C\int_0^{2\pi}\left(\int_0^3 \frac{z^3}{3}\bigg|_0^{r^2} r dr\right) d\theta = \frac{C}{3}\int_0^{2\pi}\left(\int_0^3 (r^6 - 0) r dr\right) d\theta =$$

$$= \frac{C}{3}\int_0^{2\pi}\left(\int_0^3 r^7 dr\right) d\theta == \frac{C}{3}\int_0^{2\pi} \frac{r^8}{8}\bigg|_0^3 d\theta = \frac{C}{24}\int_0^{2\pi}(3^8 - 0)d\theta = \frac{C}{24}\cdot 3^8 \cdot 2\pi = 546.75\pi C.$$

Answer: the moment of inertia is $I = 546.75\pi C$

References:

[1] George B. Thomas, Maurice D. Weir, Joel Hass, Frank R. Giordano. Thomas' Calculus 11th Edition.

[2] Area Moments of Inertia by Integration. Retrieved from www.iitg.ac.in/kd/Lecture%20Notes/ME101-Lecture18-KD.pdf