Answer on Question #66086 – Math – Calculus

Question

A torus (a dough nut-shaped object) is formed by revolving the circle x^2 + y^2 = a^2 about the vertical line x = b, where 0 < a < b. Find its volume.

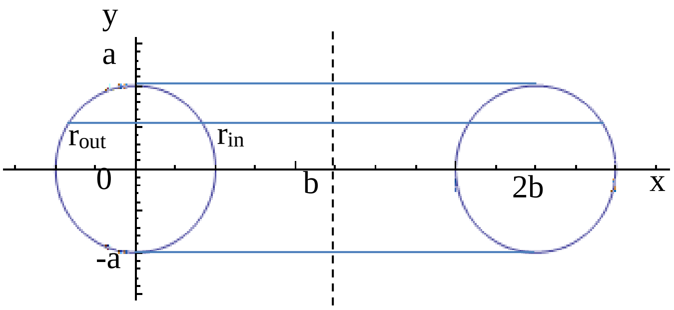
Solution

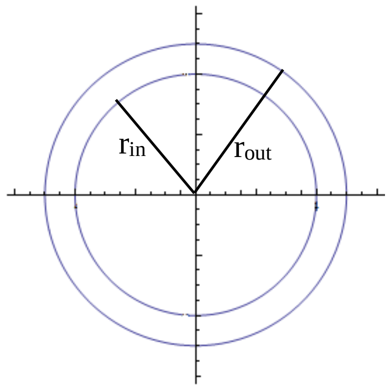
To find the volume of the torus we use the washer method [1, page 356]. A cross-section in the plane P (see figure) has the shape of a washer (an annular ring) with inner radius

$$r_{in} = b - \sqrt{a^2 - y^2}$$

and outer radius

$$r_{out} = b + \sqrt{a^2 - y^2}$$





so we find the cross-sectional area by subtracting the area of the inner circle from the area of the outer circle:

$$A(y) = \pi\left(b + \sqrt{a^2 - y^2}\right)^2 - \pi\left(b - \sqrt{a^2 - y^2}\right)^2 =$$

$$= \pi\left(b^2 + 2b\sqrt{a^2 - y^2} + a^2 - y^2\right) - \pi\left(b^2 - 2b\sqrt{a^2 - y^2} + a^2 - y^2\right) = 4\pi b\sqrt{a^2 - y^2}$$

The required volume is

$$V = \int_{-a}^{a} A(y)dy = \int_{-a}^{a} 4\pi b\sqrt{a^2 - y^2}dy = 8\pi b \int_{0}^{a} \sqrt{a^2 - y^2}dy$$

This integral can be evaluated by using a trigonometrical substitution $y = a\sin\theta$

$V = 8\pi b \int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta}\, d(a\sin\theta) = 8\pi a^2 b \int_0^{\pi/2} \cos^2\theta d\theta = 4\pi a^2 b \int_0^{\pi/2} (1 + \cos 2\theta) d\theta =$

$$4\pi a^2 b \int_0^{\pi/2} (1 + \cos 2\theta) d\theta = 4\pi a^2 b \left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2} = 2\pi^2 a^2 b$$

Answer: the volume of the torus is $V = 2\pi^2 a^2 b$.

Reference:

[1]. James Stewart. Calculus, 7th Edition.