Answer on Question #66045 – Math – Calculus

Question

Prove that the functions g(x,y) = 2x -3y / 4x+5y and h(x,y) = x/y , y≠0,y ≠ -4/5 ×x are functionally dependent

Solution

By definition, two functions $g(x,y)$ and $h(x,y)$ are said to be functionally dependent if they are functions of each other [1, page 214]. The necessary and sufficient conditions for the functional dependent of two functions is the vanishing of their Jacobian [1, page 214], i.e.

$$\begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} \\ \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} \end{vmatrix} = 0.$$

Using the Quotient Rule [2, page 132] find the partial derivatives of $g(x,y) = \frac{2x-3y}{4x+5y}$ and $h(x,y) = \frac{x}{y}$:

$$\frac{\partial g}{\partial x} = \left(\frac{2x-3y}{4x+5y}\right)_x = \frac{2(4x+5y)-4(2x-3y)}{(4x+5y)^2} = \frac{8x+10y-8x+12y}{(4x+5y)^2} = \frac{22y}{(4x+5y)^2},$$

$$\frac{\partial g}{\partial y} = \left(\frac{2x-3y}{4x+5y}\right)_y = \frac{-3(4x+5y)-5(2x-3y)}{(4x+5y)^2} = \frac{-12x-15y-10x+15y}{(4x+5y)^2} = -\frac{22x}{(4x+5y)^2},$$

$$\frac{\partial h}{\partial x} = \frac{1}{y},$$

$$\frac{\partial h}{\partial y} = -\frac{x}{y^2}.$$

Then

$$\begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} \\ \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} \end{vmatrix} = \frac{\partial g}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial h}{\partial x}\frac{\partial g}{\partial y} = \frac{22y}{(4x+5y)^2}\left(-\frac{x}{y^2}\right) - \left(-\frac{22x}{(4x+5y)^2}\right)\frac{1}{y} =$$

$$= -\frac{22x}{y(4x+5y)^2} + \frac{22x}{y(4x+5y)^2} = 0.$$

Since the Jacobian of these functions is equal to zero, then the functions $g(x,y)$ and $h(x,y)$ are functionally dependent.

Answer: functions $g(x,y)$ and $h(x,y)$ are functionally dependent.

References:

[1] S.S. Sastry. Engineering mathematics, volume two, 4th Edition

[2] James Stewart. Calculus, 7th Edition.