Answer on Question 66041 - Math - Differential Equations

Question: Solve

$$(D^3 - DD'^2 - D^2 + DD')z = 0. \qquad (1)$$

Solution: Recall that $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$. We look for the general solution $z = z(x, y)$ of the PDE with constant coefficients

$$\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial x \partial y^2} - \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = 0.$$

This operator is reducible

$$D^3 - DD'^2 - D^2 + DD' = D(D - D')(D + D' - 1).$$

Therefore the general solution of (1) has the form [1]

$$z(x, y) = u(x, y) + v(x, y) + w(x, y),$$

where u, v and w are general solutions of the first order PDEs

$$u_x = 0, \qquad v_x - v_y = 0, \qquad w_x + w_y - w = 0$$

respectively. All these PDEs can be directly solved by the Lagrange method:

$$u_x = 0 \quad \Rightarrow \quad u(x, y) = f(y);$$

$$v_x - v_y = 0 \quad \Rightarrow \quad dx = -dy \quad \Rightarrow \quad x + y = c \quad \Rightarrow \quad v(x, y) = g(x + y);$$

$$w_x + w_y = w \quad \Rightarrow \quad dx = dy = \frac{dw}{w} \quad \Rightarrow \quad x - y = c_1,\ we^{-x} = c_2$$

$$\Rightarrow \quad we^{-x} = h(x - y) \quad \Rightarrow \quad w = e^x h(x - y),$$

where f, g and h are arbitrary C^1-functions. Finally,

$$z(x, y) = f(y) + g(x + y) + e^x h(x - y).$$

Answer: $z(x, y) = f(y) + g(x + y) + e^x h(x - y)$, where f, g and h are arbitrary C^1-functions.

References

[1] Sneddon, Ian N., and J. C. Polkinghorne. Elements of partial differential equations. Physics Today 10.5 (1957): 96–109.