Answer on Question #66013 – Math – Calculus

Question

Apply Inverse function theorem to check the local invertibility of the following function $f: R^2 \to R^2$ given by $f(x, y) = (y\, cosx, x - y + 2)$ at the point (0,π).

Solution

We have the vector-valued function f from R^2 to R^2. The Jacobian matrix is

$$J_f(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial(y\cos x)}{\partial x} & \frac{\partial(y\cos x)}{\partial y} \\ \frac{\partial(x - y + 2)}{\partial x} & \frac{\partial(x - y + 2)}{\partial y} \end{bmatrix} = \begin{bmatrix} -ysin\, x & \cos x \\ 1 & -1 \end{bmatrix}$$

and the determinant is

$$\det J_f(x, y) = \begin{vmatrix} -y\, sinx & \cos x \\ 1 & -1 \end{vmatrix} = ysin\, x - \cos x.$$

We see that f is continuously differentiable at every point from R^2.

The value of the determinant at the given point (0,π) is

$$\det J_f(0, \pi) = \pi sin\, 0 - \cos 0 = -1 \neq 0.$$

So, the determinant is nonzero at the point (0,π). By Inverse function theorem, for the point (0,π), there exists a neighborhood over which f is invertible.

The statement of the Inverse function theorem is given below.

The Inverse Function Theorem [1].

Let $f: R^n \to R^n$ be continuously differentiable on some open set containing a, and suppose $\det J_f(a) \neq 0$. Then there is some open set V containing a and an open W containing $f(a)$ such that $f: V \to W$ has a continuous inverse $f^{-1}: W \to V$ which is differentiable for all $y \in W$.

Answer: The function f is invertible near the point (0,π).

Reference:

[1] The Inverse Function Theorem. Retrieved from

http://www.math.ucsd.edu/~nwallach/inverse[1].pdf