Answer on Question #66010 – Math – Calculus

Question

Find the moment of inertia I2 for the solid above the xy-plane bounded by the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 9$ assuming the mean density to be constant C.

Solution

$$I_{xOy} = \iiint z^2 \rho(x, y, z) dx dy dz,$$

where

$$\rho(x, y, z) = C.$$

Using substitution

$$\begin{cases} x = r cos\varphi \\ y = r sin\varphi \\ z = z \end{cases}$$

we get

$x^2 + y^2 = 9 \implies r^2 = 9$;

$0 \le z \le r^2$;

$0 \le r \le 3$;

$0 \le \varphi \le 2\pi$.

The moment of inertia for the solid above the xy-plane is given by

$$I_{xOy} = C \int_0^{2\pi} d\varphi \int_0^3 r \left(\int_0^{r^2} z^2 \, dz \right) dr = C \cdot \varphi|_0^{2\pi} \int_0^3 r \frac{z^3}{3}\Big|_0^{r^2} dr =$$

$$= 2\pi C \cdot \frac{1}{3} \int_0^3 r^7 \, dr = 2\pi C \frac{r^8}{3 \cdot 8}\Big|_0^3 = \frac{2\pi C}{3 \cdot 8}(3^8 - 0^8) = \frac{2187\pi C}{4}.$$

Answer: $I_{xOy} = \frac{2187\pi C}{4}$.