Answer on Question #65997 – Math – Calculus

Question

Find the centre of gravity of a mass in the shape of a semicircular disc of radius 4, if the density at (x, y) is (2y)/(x^2+y^2)

Solution

If

$$\rho(x,y)=\frac{2y}{x^2+y^2}$$

is the density of a semicircular disc at $(x,\ y)$ and region is given by

$$R=\{(x,y): x^2+y^2\le 4^2, y\ge 0\},$$

then mass is

$$m=\iint\limits_R \rho(x,y)dxdy=\int\limits_0^{\pi} d\varphi\int\limits_0^4 \frac{2r\sin\varphi}{r^2}rdr=\int\limits_0^{\pi} d\varphi\int\limits_0^4 2\sin\varphi dr=8\int\limits_0^{\pi} s\sin\varphi d\varphi=-8(\cos\pi-\cos 0)=$$

$$=-8(\cos\pi-\cos 0)=-8(-1-1)=16;$$

the centre of gravity of a mass has coordinates as follows:

$$x_0=\frac{\iint\limits_R x\rho(x,y)dxdy}{m},\quad y_0=\frac{\iint\limits_R y\rho(x,y)dxdy}{m}.$$

Then

$$x_0=\frac{\iint\limits_R x\rho(x,y)dxdy}{m}=\frac{1}{16}\iint\limits_R \frac{2xy}{x^2+y^2}dxdy=\frac{1}{16}\int\limits_0^{\pi} d\varphi\int\limits_0^4 \frac{2r^2\cos\varphi\sin\varphi}{r^2}rdr=\frac{1}{16}\int\limits_0^{\pi} d\varphi\int\limits_0^4 r\sin 2\varphi dr=$$

$$=\frac{1}{16}\int\limits_0^{\pi}\sin 2\varphi d\varphi\int\limits_0^4 rdr=-\frac{1}{16}\frac{1}{2}\cos 2\varphi\Big|_0^{\pi}\cdot\frac{r^2}{2}\Big|_0^4=-\frac{1}{64}(\cos 2\pi-\cos 0)\cdot(4^2-0)=-\frac{1}{64}(1-1)\cdot(16-0)=0$$

$$y_0=\frac{\iint\limits_R y\rho(x,y)dxdy}{m}=\frac{1}{16}\iint\limits_R \frac{2y^2}{x^2+y^2}dxdy=\frac{1}{8}\int\limits_0^{\pi} d\varphi\int\limits_0^4 \frac{r^2\sin^2\varphi}{r^2}rdr=\frac{1}{8}\int\limits_0^{\pi} d\varphi\int\limits_0^4 r\sin^2\varphi dr=$$

$$=\frac{1}{8}\int\limits_0^{\pi}\sin^2\varphi d\varphi\int\limits_0^4 rdr=\frac{1}{8}\int\limits_0^{\pi}\frac{1}{2}(1-\cos 2\varphi)d\varphi\int\limits_0^4 rdr=\frac{1}{16}(\varphi-\frac{1}{2}\sin 2\varphi)\Big|_0^{\pi}\cdot\frac{r^2}{2}\Big|_0^4=$$

$$=\frac{1}{32}(\pi-\frac{1}{2}\sin 2\pi-0+\frac{1}{2}\sin 0)\cdot(4^2-0)=\frac{1}{32}\pi\cdot 16=\frac{\pi}{2}.$$

Answer: the centre of gravity of a mass in the shape of a semicircular disc of radius 4 is $x_0=0$, $y_0=\frac{\pi}{2}$.