Answer on Question #65799 – Math – Calculus

Question

Find the points (x,y) on the unit circle, at which the product xy is maximum or minimum

Solution

We write this product in polar coordinates. The following formula connects polar and cartesian coordinates [1]:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Then for product xy we get

$$xy = r^2\sin\theta\cos\theta$$

Given the points lie on the unit circle, $r = 1$, hence this product is

$$xy = \sin\theta\cos\theta$$

Thus this product is a function of the angular variable θ with domain $0 \le \theta \le 2\pi$

$$xy = f(\theta) = \sin\theta\cos\theta$$

To find maximum or minimum of this function we use Fermat's Theorem [1]: If f has a local maximum or minimum at $\theta = \theta_0$, then θ_0 is a critical number of f.

Find critical numbers of $f(\theta)$:

$$f'(\theta) = (\sin\theta\cos\theta)' = (\sin\theta)'\cos\theta + \sin\theta(\cos\theta)' = \cos\theta \cdot \cos\theta + \sin\theta \cdot (-\sin\theta) =$$
$$= \cos^2\theta - \sin^2\theta = 0.$$

Dividing by $\cos^2\theta$ we get

$$\frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta = 1 \quad \Rightarrow \quad \tan\theta = \pm 1$$

This equation has solutions:

1. $\theta_1 = \frac{\pi}{4}$ in the first quadrant
2. $\theta_1 = \frac{3\pi}{4}$ in the second quadrant
3. $\theta_3 = \frac{5\pi}{4}$ in the third quadrant
4. $\theta_4 = \frac{7\pi}{4}$ in the fourth quadrant

Thus $\theta_1,\ \theta_2, \theta_3, \theta_4$ are the critical numbers of the function $xy = f(\theta) = \sin\theta\cos\theta$.

For each case now we write x and y:

1. $x_1 = 1 \cdot \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\ y_1 = 1 \cdot \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ so that $(x_1, y_1) = \left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$ is the maximum point, here $x_1 \cdot y_1 = \frac{1}{2}$.
2. $x_2 = 1 \cdot \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$, $\ y_2 = 1 \cdot \sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$ so that $(x_2, y_2) = \left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$ is the minimum point, here $x_2 \cdot y_2 = -\frac{1}{2}$.
3. $x_3 = 1 \cdot \cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$, $\ y_3 = 1 \cdot \sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$ so that $(x_3, y_3) = \left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$ is the maximum point, here $x_3 \cdot y_3 = \frac{1}{2}$.
4. $x_4 = 1 \cdot \cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2}$, $\ y_4 = 1 \cdot \sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$ so that $(x_4, y_4) = \left(\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$ is the minimum point, here $x_4 \cdot y_4 = -\frac{1}{2}$.

Answer: the points (x,y) on the unit circle at which the product xy is maximum or minimum are as follows:

$(x_1, y_1) = \left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$ is the maximum point

$(x_2, y_2) = \left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$ is the minimum point

$(x_3, y_3) = \left(-\frac{\sqrt{2}}{2},\ \ -\frac{\sqrt{2}}{2}\right)$ is the maximum point
$(x_4, y_4) = \left(\frac{\sqrt{2}}{2},\ \ -\frac{\sqrt{2}}{2}\right)$ is the minimum point.

References:

[1] James Stewart. Calculus.