Answer on Question #65794 – Math – Calculus

Question

Using polar coordinates, show that $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^3-y^3}{x^2+y^2} = 0$. Also, find the two repeated limits.

Solution

Let us introduce the following change of variables

(see https://en.wikipedia.org/wiki/Polar_coordinate_system):

$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$, where $r \to 0$, and φ is a fixed value. Then

$\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^3-y^3}{x^2+y^2} = \lim\limits_{r\to 0} \frac{r^3\left((\cos\varphi)^3-(\sin\varphi)^3\right)}{r^2\left((\cos\varphi)^2+(\sin\varphi)^2\right)} = \left((\cos\varphi)^3 - (\sin\varphi)^3\right)\lim\limits_{r\to 0} r = 0$ for all possible values of φ $\left((\cos\varphi)^2 + (\sin\varphi)^2 = 1\right.$ due to the well-known trigonometric identity, see http://www.math.com/tables/trig/identities.htm). So the assertion is proved.

Let us calculate two repeated limits (see https://en.wikipedia.org/wiki/Iterated_limit):

$\lim\limits_{x\to 0}\lim\limits_{y\to 0} \frac{x^3-y^3}{x^2+y^2} = \lim\limits_{x\to 0} \frac{x^3}{x^2} = \lim\limits_{x\to 0} x = 0;$

$\lim\limits_{y\to 0}\lim\limits_{x\to 0} \frac{x^3-y^3}{x^2+y^2} = \lim\limits_{y\to 0} \frac{-y^3}{y^2} = -\lim\limits_{y\to 0} y = 0.$

Answer: $\lim\limits_{x\to 0}\lim\limits_{y\to 0} \frac{x^3-y^3}{x^2+y^2} = \lim\limits_{y\to 0}\lim\limits_{x\to 0} \frac{x^3-y^3}{x^2+y^2} = \lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^3-y^3}{x^2+y^2} = 0.$