Answer on Question #65646 – Math – Calculus

Question

Calculate

$$\lim_{x\to 0}\left(\frac{3^x}{x}-\frac{2^x}{x}\right).$$

Solution

Let us prove the very important formula of limit which will be later used in the question. So, we will prove that

$$\lim_{x\to 0}\frac{a^x-1}{x}=\ln a \quad \text{for any} \quad a>0.$$

Let $y=a^x-1$, then $y>-1$ for all real x and $1+y=a^x$.
Taking logarithm on both sides (which are both positive), we have

$$\ln(1+y)=\ln a^x \Rightarrow \ln(1+y)=x\ln a \Rightarrow x=\frac{\ln(1+y)}{\ln a}.$$

Also $\lim_{x\to 0} y=\lim_{x\to 0}(a^x-1)=a^0-1=1-1=0.$
This shows that $y\to 0$ as $x\to 0$. Therefore, the initial limit can be rewritten as

$$\lim_{x\to 0}\frac{a^x-1}{x}=\lim_{y\to 0}\frac{y}{\frac{\ln(1+y)}{\ln a}}=\lim_{y\to 0}\frac{\ln a}{\frac{1}{y}\ln(1+y)}=\lim_{y\to 0}\frac{\ln a}{\ln(1+y)^{1/y}}=\frac{\ln a}{\ln\left(\lim_{y\to 0}(1+y)^{1/y}\right)}.$$

Using the relation $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$, we have

$$\lim_{x\to 0}\frac{a^x-1}{x}=\frac{\ln a}{\ln e}=\ln a.$$

Let us now calculate the limit in the question:

$$\lim_{x\to 0}\left(\frac{3^x}{x}-\frac{2^x}{x}\right)=\lim_{x\to 0}\left(\frac{3^x-1}{x}-\frac{2^x-1}{x}\right)=\lim_{x\to 0}\frac{3^x-1}{x}-\lim_{x\to 0}\frac{2^x-1}{x}=\ln 3-\ln 2=\ln\frac{3}{2}=\ln 1.5$$

Answer: $\ln 1.5$.