Answer on Question #65108 – Math – Statistics and Probability

Question

For the random variable X with the following probability density function

$$f(x) = \begin{cases} 0, & x < 0, \\ \frac{x}{2}, & 0 \le x \le 2, \\ 0, & x > 2. \end{cases}$$

i) Find $P(|X - E(X)| > 2)$;

ii) Use Chebyshev's inequality to obtain an upper bound on $P(|X - E(X)| > 2)$ and compare with the result in i).

Solution

i) By formula (45) on p. 235 in [1]

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx = \int_0^2 x \cdot \frac{x}{2} dx = \left.\frac{x^3}{6}\right|_0^2 = \frac{4}{3}.$$

Then

$$P(|X - E(X)| > 2) = P\left(\left|X - \frac{4}{3}\right| > 2\right)$$

$$= P\left(-\infty < X - \frac{4}{3} < -2\right) + P\left(2 < X - \frac{4}{3} < +\infty\right) =$$

$$P\left(-\infty < X < -\frac{2}{3}\right) + P\left(\frac{10}{3} < X < +\infty\right) =$$

$$= \int_{-\infty}^{-2/3} f(x)dx + \int_{10/3}^{+\infty} f(x)dx = 0.$$

ii) We use Chebyshev's inequality ([1], eq. (23) on p. 229):

$$P(|X - E(X)| > 2) \le \frac{E(X - E(X))^2}{2^2}.$$

We have

$$E(X - E(X))^2 = E(X)^2 - \big(E(X)\big)^2.$$

Applying formula (45) on p. 235 in [1] again, we get

$$E(X - E(X))^2 = E(X)^2 - \big(E(X)\big)^2 = \int_0^2 x^2 \cdot \frac{x}{2} dx - \left(\frac{4}{3}\right)^2 = \left.\frac{x^4}{8}\right|_0^2 - \frac{16}{9} = 2 - \frac{16}{9} = \frac{2}{9}.$$

Then

$$P(|X - E(X)| > 2) \le \frac{2}{9 \cdot 4} = \frac{1}{18} = 0.05555556.$$

Answer: i) 0, **ii)** 0.05555556. The probability estimated with Chebyshev's inequality in ii) is small but it does not equal 0 as in i).

References:

[1]. A. N. Shiryaev. Probability-1. Third edition. Springer, 2016.