Answer on Question #64854 – Math – Linear Algebra

Question

Find the orthogonal canonical reduction of the quadratic form

$$-x^2 + y^2 + z^2 + 2xy - 2xz + 2yz$$

Also, find its principal axes.

Solution

The matrix of the quadratic form:

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} -1-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ -1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ -1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1-\lambda \\ -1 & 1 \end{vmatrix} = 0$$

$$(-1-\lambda)((1-\lambda)^2 - 1) - (1-\lambda+1) - (1+1-\lambda) = 0$$

$$2\lambda + 2\lambda^2 - \lambda^2 - \lambda^3 + 2\lambda - 4 = 0$$

$$\lambda^3 - \lambda^2 - 4\lambda + 4 = 0$$

$$\lambda^2(\lambda - 1) - 4(\lambda - 1) = 0$$

$$(\lambda^2 - 4)(\lambda - 1) = 0$$

$$\lambda_1 = 1\,;\ \lambda_2 = -2\,;\ \lambda_3 = 2$$

The orthogonal canonical reduction:

$$Q = \lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2$$

$$Q = x'^2 - 2y'^2 + 2z'^2$$

For $\lambda_1 = 1$:

$$\begin{pmatrix} -1-1 & 1 & -1 \\ 1 & 1-1 & 1 \\ -1 & 1 & 1-1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

The principal axis:

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = -2$:

$$\begin{pmatrix} -1+2 & 1 & -1 \\ 1 & 1+2 & 1 \\ -1 & 1 & 1+2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

The principal axis:

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 2$:

$$\begin{pmatrix} -1-2 & 1 & -1 \\ 1 & 1-2 & 1 \\ -1 & 1 & 1-2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

The principal axis:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Answer: $x'^2 - 2y'^2 + 2z'^2$; $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$