Answer on Question #64521 – Math – Real Analysis

Question

For x_n given by the following formulas, establish either the convergence or divergence of the sequence

(a) $x_n = \frac{n}{n+1}$, **(b)** $x_n = \frac{(-1)^n n}{n+1}$, **(c)** $x_n = \frac{n^2}{n+1}$, **(d)** $x_n = \frac{2n^2+3}{n^2+1}$

Solution (a). Let's consider the sequence $x_n = \frac{n}{n+1}$, $n \geq 1$. Its limit is

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n} \frac{1}{1+\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

Thus, the sequence $x_n = \frac{n}{n+1}$ is convergent.

Answer: (a) $x_n = \frac{n}{n+1}$ is convergent.

Solution (b). Let's consider the subsequences $x_{2k} = \frac{(-1)^{2k} 2k}{2k+1} = \frac{2k}{2k+1} = 1 - \frac{1}{2k+1}$ and $x_{2k-1} = \frac{(-1)^{2k-1}(2k-1)}{2k-1+1} = \frac{1-2k}{2k} = \frac{1}{2k} - 1$ of the sequence $x_n = \frac{(-1)^n n}{n+1}$.

Since $\lim_{k\to\infty} x_{2k} = \lim_{k\to\infty}\left(1 - \frac{1}{2k+1}\right) = 1$ and $\lim_{k\to\infty} x_{2k-1} = \lim_{k\to\infty}\left(\frac{1}{2k} - 1\right) = -1$, then the sequence $x_n = \frac{(-1)^n n}{n+1}$ has two different limit points. Therefore $x_n = \frac{(-1)^n n}{n+1}$ is divergent.

Answer: (b) $x_n = \frac{(-1)^n n}{n+1}$ is divergent.

Solution (c). Since $x_n = \frac{n^2}{n+1} = \frac{n^2-1+1}{n+1} = n-1+\frac{1}{n+1} > n-1$ for all natural n and sequence $y_n = n-1$ is unbounded from above. Hence $x_n = \frac{n^2}{n+1}$ is unbounded, $x_n = \frac{n^2}{n+1}$ is divergent.

Answer: (c) $x_n = \frac{n^2}{n+1}$ is divergent.

Solution (d). Let's consider the sequence $x_n = \frac{2n^2+3}{n^2+1}$, $n \geq 1$. Its limit is

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{2n^2+3}{n^2+1} = \lim_{n\to\infty} \frac{n^2}{n^2} \frac{2+\frac{3}{n^2}}{1+\frac{1}{n}} = \lim_{n\to\infty} \frac{2+\frac{3}{n^2}}{1+\frac{1}{n}} = 2.$$

So, the sequence $x_n = \frac{2n^2+3}{n^2+1}$ is convergent.

Answer: (d) $x_n = \frac{2n^2+3}{n^2+1}$ is convergent.