Answer on Question #64390 – Math – Linear Algebra

Question

Q. Show that the transformation between the coordinates X1, X2, X3 and X1' , X2' , X3' defined by
X1'=1/3(2x1+2x2-x3)
X2'=1/3(2x1-x2+2x3)
X3'=1/3(-x1+2x2+2x3)
Is orthogonal and left-handed(improper)

Solution

For the matrix

$$A = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix},$$

the dot products (scalar products) of the different rows (the sum of the products of items) are as follows:

for 1st and 2nd rows, $\frac{1}{3}(2 \cdot 2 + 2 \cdot (-1) + (-1) \cdot 2) = 0;$

for 1st and 3rd rows, $\frac{1}{3}(2 \cdot (-1) + 2 \cdot 2 + (-1) \cdot 2) = 0;$

for 2nd and 3rd rows, $\frac{1}{3}(2 \cdot (-1) + (-1) \cdot 2 + 2 \cdot 2) = 0.$

The scalar products of each row by itself (the sum of the squares) are as follows:

for 1st row, $\frac{1}{9}(2^2 + 2^2 + (-1)^2) = 1;$

for 2nd row, $\frac{1}{9}(2^2 + (-1)^2 + 2^2) = 1;$

for 3rd row, $\frac{1}{9}((-1)^2 + 2^2 + 2^2) = 1.$

This means that

$$AA^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E,$$

i.e, A is an orthogonal matrix.

Next,

$$Det\ A = \frac{1}{3^3}\big(2 \cdot (-1) \cdot 2 + 2 \cdot 2 \cdot (-1) + (-1) \cdot 2 \cdot 2 - 2 \cdot 2 \cdot 2 - 2 \cdot 2 \cdot 2 - (-1) \cdot (-1) \cdot (-1)\big) = -1$$

If $Det\ A = -1$, then it means that A is improper.

QED.