Answer on Question #64381 – Math – Linear Algebra

Question

Show that the transformation between the coordinates x_1, x_2, x_3 and x_1', x_2', x_3' defined by

$$x_1' = \frac{1}{3}(2x_1 + 2x_2 - x_3),$$

$$x_2' = \frac{1}{3}(2x_1 - x_2 + 2x_3),$$

$$x_3' = \frac{1}{3}(-x_1 + 2x_2 + 2x_3)$$

is orthogonal and left-handed (improper).

Solution

For the matrix

$$A = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix},$$

the dot products (scalar products) of the different rows will be the sum of the products of items, they are as follows:

for 1st and 2nd rows,

$$\frac{1}{3}(2 \cdot 2 + 2 \cdot (-1) + (-1) \cdot 2) = 0;$$

for 1st and 3rd rows,

$$\frac{1}{3}(2 \cdot (-1) + 2 \cdot 2 + (-1) \cdot 2) = 0;$$

for 2nd and 3rd rows,

$$\frac{1}{3}(2 \cdot (-1) + (-1) \cdot 2 + 2 \cdot 2) = 0.$$

The scalar products of each row by itself will be the sum of the squares, they are as follows:

for 1st and 2nd rows,

$$\frac{1}{9}(2^2 + 2^2 + (-1)^2) = 1;$$

for 1st and 3rd rows,

$$\frac{1}{9}(2^2 + (-1)^2 + 2^2) = 1;$$

for 2nd and 3rd rows,

$$\frac{1}{9}((-1)^2 + 2^2 + 2^2) = 1.$$

This means that

$$AA^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E,$$

i.e., A is an orthogonal matrix.

Next,

$$\det A = \begin{vmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{vmatrix} =$$

$$= \frac{1}{3^3}\big(2\cdot(-1)\cdot 2 + 2\cdot 2\cdot(-1) + (-1)\cdot 2\cdot 2 - 2\cdot 2\cdot 2 - 2\cdot 2\cdot 2 - (-1)\cdot$$
$$\cdot(-1)\cdot(-1)\big) = -1$$

If $\det A = -1$, then it means that A is improper.

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