Answer on Question #64245 – Math – Differential Equations

Question

Taylor's expansion of $f(x) = Sin\ x$ at $x = 0$.

Solution

By the definition, Taylor expansion is given by

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$,

where a is a point where the expansion is evaluated,

$n! = 1 \cdot 2 \cdot \cdots \cdot n$ is the factorial of n,

$f^{(n)}(a)$ is the nth derivative of f evaluated at the point a.

In this problem

$f(x) = \sin(x), f^{(n)}(a) = \sin\left(a + \frac{n\pi}{2}\right), a = 0,$

$$f^{(n)}(0) = \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 1, if\ n = 4m + 1, m\ is\ integer, \\ 0,\ if\ n = 2m, \quad\quad m\ is\ integer, \\ -1, if\ n = 4m + 3, m\ is\ integer. \end{cases}$$

In other words, the derivatives of even degree are zero.

General formula of expansion of $f(x) = \sin x$ at $x = 0$:

$$\sin\ x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Answer: $\sin\ x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$