Answer on Question #63947 – Math – Geometry

Question

A manufacturer makes aluminum cups of a given volume (16 cubic inches) in the form of right circular cylinders open at the top. Find the dimensions which use the least material.

Solution

Let R be the radius of right circular cylinder, and H be its height.

Then the volume is $V = \pi R^2 H = 16$, hence $H = \frac{16}{\pi R^2}$.

The area of the base is $S_{base} = \pi R^2$.

The lateral area of the cylinder is $S_{side} = 2\pi R H$.

Total surface area of the cup is $S = S_{base} + S_{side} = \pi R^2 + 2\pi R H$.

Substituting $H = \frac{16}{\pi R^2}$ into the formula for the total surface area

$$S = \pi R^2 + 2\pi R \frac{16}{\pi R^2} = \pi R^2 + \frac{32}{R}.$$

It's a function of variable R. To determine the minimum first we'll find a point at which its derivative is equal to zero:

$$S' = \left(\pi R^2 + \frac{32}{R}\right)' = 2\pi R - \frac{32}{R^2} = 0 \implies 2\pi R = \frac{32}{R^2} \implies R^3 = \frac{16}{\pi} \implies$$

$$R = \sqrt[3]{\frac{16}{\pi}} = 2 \cdot \sqrt[3]{\frac{2}{\pi}} \text{ and } S''(R) = (S')'(R) = \left(2\pi R - \frac{32}{R^2}\right)'(R) =$$

$$= (2\pi - 32 \cdot (-2) R^{-3})(R) = \left(2\pi + \frac{64}{R^3}\right)(R) = 2\pi + \frac{64}{16/\pi} > 0, \text{ therefore}$$

$R = 2 \cdot \sqrt[3]{\frac{2}{\pi}}$ is a minimum indeed and finally

$$H_{min} = \frac{16}{\pi R^2} = \frac{16}{\pi\left(2 \cdot \sqrt[3]{\frac{2}{\pi}}\right)^2} = \frac{4}{\pi\left(\frac{2}{\pi}\right)^{2/3}} = 2 \cdot \frac{2}{\pi} \cdot \left(\frac{2}{\pi}\right)^{-2/3} = 2 \cdot \left(\frac{2}{\pi}\right)^{1/3} = 2 \cdot \sqrt[3]{\frac{2}{\pi}} \approx 1.72\ in.$$

Answer: The radius of the cup's bottom and its height are both equal to $2 \cdot \sqrt[3]{\frac{2}{\pi}} \approx 1.72\ in.$