Answer on Question #63944 – Math – Geometry

Question

A right triangle has hypotenuse of length 13 and one leg of length 5. Find the dimensions of the rectangle of largest area which has one side along the hypotenuse and the ends of the opposite side on the legs of this triangle?

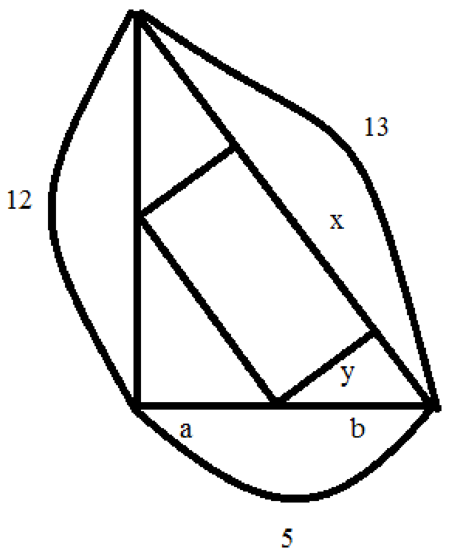
Solution

The right triangle has hypotenuse of length 13 and one leg of length 5.

Using the Pythagorean Theorem the other leg is

$\sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12.$

The rectangle creates three small triangles inside the big triangle. These are all similar to the big triangle by Angle-Angle Similarity. If we look at the leg of length 5, it's now split into 2 parts of length a and b.



A small triangle with the leg a and the hypotenuse x is s similar to the big triangle with legs 5, 12 and the hypotenuse 13, hence

$$\frac{13}{x} = \frac{5}{a} \rightarrow a = \frac{5x}{13}$$

A small triangle with the leg y and the hypotenuse b is s similar to the big triangle with legs 5, 12 and the hypotenuse 13, hence

$$\frac{12}{y}=\frac{13}{b}\rightarrow b=\frac{13y}{12}.$$

Next,

$$a+b=5.$$

Using all previous formulae obtain

$\frac{5x}{13}+\frac{13y}{12}=5;$

$y=\frac{12}{13}\left(5-\frac{5x}{13}\right);$

$y=\frac{780-60x}{169};$

The area of the rectangle is

$S=xy=x\cdot\frac{780-60x}{169}=\frac{780}{169}x-\frac{60}{169}x^2.$

Then

$S'=\frac{780}{169}-\frac{120}{169}x=\frac{780-120x}{169};$

$S'=0\rightarrow \frac{780-120x}{169}=0\rightarrow\frac{120x}{169}=\frac{780}{169}\rightarrow x=\frac{780}{120}=\frac{13}{2}\rightarrow x=6.5;$

$y=\frac{780-60x}{169}=\frac{780-60\cdot 6.5}{169}=\frac{390}{169}=\frac{30}{13}\approx 2.31;$

$S''=(S')'=\left(\frac{780-120x}{169}\right)'=-\frac{120}{169}<0$, hence the function S has a local maximum at the point $x=6.5$.

Values $x>0$ and $x<13$ are under consideration.

Thus, the largest area of the rectangle is

$$S_{max}=xy=\frac{13}{2}\cdot\frac{30}{13}=\frac{30}{2}=15$$

Answer: 6.5 and $\frac{30}{13}$.