Answer on Question #63943 – Math – Geometry

Question

A box with square base is to have an open top. The area of the material in the box is to be 100 in square. What should the dimensions be in order to make the volume as large as possible?

Solution

Let the volume, the length of the bottom side, and the height be V, x, and y respectively.
Then the area of each of 4 lateral faces of the box is xy, the area of the bottom is x^2.
Thus,

$$V = x^2 y,\ x^2 + 4xy = 100, x > 0\ , y > 0.$$

Hence $y = \frac{100-x^2}{4x}$, $V = x^2 y = \frac{x^2(100-x^2)}{4x} = 25x - x^3/4$.
Assume that $x > 0$.
Thus, the maximum of the function $V(x) = 25x - x^3/4$ should be found.
To find the maximum, set the derivative to zero:

$V'(x) = (25x - x^3/4)' = 25 - 3x^2/4 = 0$, hence $x = \frac{10}{\sqrt{3}} \approx 5.77$,

$$y = \frac{100-\left(\frac{10}{\sqrt{3}}\right)^2}{4\cdot\frac{10}{\sqrt{3}}} = \frac{5}{\sqrt{3}} \approx 2.89;$$

$$V''(x) = (V'(x))' = (25 - 3x^2/4)' = -\frac{6x}{4} = -\frac{3x}{2},$$

$V''\left(\frac{10}{\sqrt{3}}\right) = -\frac{3}{2}\cdot\frac{10}{\sqrt{3}} = -5\sqrt{3} < 0$, hence $x = \frac{10}{\sqrt{3}},\ y = \frac{5}{\sqrt{3}}$ indeed make the volume as large as possible.

Answer:

The volume is as large as possible if the length of the side of the bottom is $\frac{10}{\sqrt{3}} \approx 5.77$, the height of the box is $\frac{5}{\sqrt{3}} \approx 2.89$.