Answer on Question #63839 – Math - Real Analysis

Question

Prove that set of reals $\mathbb{R}$ is an ordered field.

Solution

Axioms of addition. For any two real numbers a, b, there is an operation of addition which associates their sum denoted by $a + b$. The operation of addition satisfies the following axioms:

A1. (Associativity) $(x + y) + z = x + (y + z)$.

A2. (Existence of zero) There is a real number, called zero and denoted by 0, such that

$x + 0 = 0 + x = x$ for all real numbers x.

A3. (Existence of negative) For every $x \in \mathbb{R}$ there is $y \in \mathbb{R}$ such that

$x + y = y + x = 0$. The number *y* is called the negative of *x* and denoted by −*x*.

A4. (Commutativity) $x + y = y + x$.

Hence $\mathbb{R}$ with the operation of addition is a commutative group.

Axioms of multiplication. There is an operation of multiplication which associates with any two real numbers *x*, *y*, the number $x \cdot y$. It satisfies the following axioms:

M1. (Associativity) $(x \cdot y) \cdot = x \cdot (y \cdot z)$.

M2. (Existence of identity) There is a real number, called identity and denoted by 1, such that $1 \neq 0$ and $1 \cdot x = x \cdot 1 = x$ for all real numbers x.

M3. (Existence of reciprocal) For every $x \in \mathbb{R} \setminus \{0\}$ there is $y \in \mathbb{R}$ such that $x \cdot y = y \cdot x = 1$. (the number y is called the reciprocal of x and denoted by x −1 or 1 x).

M4. (Commutativity) $x \cdot y = y \cdot x$.

Hence $\mathbb{R} \setminus \{0\}$ together with the multiplication is a commutative group.

The distributive law. The next axiom defines the relation between the two operations.

D. $x \cdot (y + z) = x \cdot y + x \cdot z$

All computational rules for real numbers follow from the axioms A1–A4, M1-M4, D.

Order There is a subset P of $\mathbb{R}$, called the set of positive numbers, satisfying the following two exioms:

O1. If $x, y \in P$, then $x + y$ and $x \cdot y \in P$.

O2. For every $x \in \mathbb{R}$, either $x \in P$ or $x = 0$ or $-x \in P$.

The axioms O1 and O2 imply that 1 is a positive number. Indeed, since $1 \neq 0$, the axiom O1 implies that either 1 or −1 is positive. Since $1 = 1 \cdot 1 = (-1) \cdot (-1)$, the axiom O1 implies that 1 is positive.

A set X with two operations, addition and multiplication, which satisfy axioms A1–A4, M1-M4, D, and O1-O2 is called an ordered field. The set of real numbers $\mathbb{R}$ is an ordered field.