Answer on Question #63549 – Math – Algebra

Question

If α and β are the roots of the equation $x^2 + 2x + p = 0$ and if $\frac{1-\alpha}{\alpha}$ and $\frac{1-\beta}{\beta}$ are the root s of the equation $4x^2 - 3x + q = 0$ find p and q.

Solution

Applying Vieta's formulas to the equation $x^2 + 2x + p = 0$ we get

$$\begin{matrix} \alpha + \beta = -2 \\ \alpha \cdot \beta = p \end{matrix} \quad \textbf{(1)}$$

Applying Vieta's formulas to the equation $4x^2 - 3x + q = 0$ we get

$$\begin{matrix} \frac{1-\alpha}{\alpha} + \frac{1-\beta}{\beta} = \frac{3}{4} \\ \frac{1-\alpha}{\alpha} \cdot \frac{1-\beta}{\beta} = \frac{q}{4}. \end{matrix} \quad \textbf{(2)}$$

It follows from the second equation of (2) that

$$\frac{(1-\alpha)(1-\beta)}{\alpha\beta} = \frac{q}{4},$$

$$\frac{1-\beta-\alpha+\alpha\beta}{\alpha\beta} = \frac{q}{4},$$

$$\frac{1-(\alpha+\beta)+\alpha\beta}{\alpha\beta} = \frac{q}{4},$$

$$q = 4 \cdot \frac{1-(\alpha+\beta)+\alpha\beta}{\alpha\beta} \quad \textbf{(3)}$$

Method 1

It follows from the first equation of (2) that

$$\frac{1}{\alpha} - 1 + \frac{1}{\beta} - 1 = \frac{3}{4},$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = 2 + \frac{3}{4},$$

$$\frac{\beta+\alpha}{\alpha\beta} = \frac{2\cdot 4+3}{4},$$

$$\frac{\beta+\alpha}{\alpha\beta} = \frac{11}{4}. \quad \textbf{(4)}$$

Substituting $\alpha + \beta = -2$, $\alpha \cdot \beta = p$ from (1) into (4) we get

$$\frac{-2}{p} = \frac{11}{4},$$

hence

$$p = \frac{-2 \cdot 4}{11} = -\frac{8}{11} \approx -0.727.$$

Thus,

$$\alpha\beta = p = \frac{-2 \cdot 4}{11} = -\frac{8}{11} \approx -0.727. \quad \textbf{(5)}$$

Substituting $\alpha + \beta = -2$ from (1), $\alpha\beta = -\frac{8}{11}$ from (5) into (3) we get

$$q = 4\frac{1-(-2)+\left(-\frac{8}{11}\right)}{\left(-\frac{8}{11}\right)} = 4 \cdot \frac{3-\frac{8}{11}}{-\frac{8}{11}} = 4 \cdot \frac{\frac{3 \cdot 11 - 8}{11}}{-\frac{8}{11}} = 4 \cdot \frac{\frac{25}{11}}{-\frac{8}{11}} = 4 \cdot \left(-\frac{25}{8}\right) = -12.5.$$

Method 2

It follows from the first equation (1) that $\alpha = -2 - \beta$.

Substituting for α into the first equation of (2) obtain

$$\frac{1-(-2-\beta)}{(-2-\beta)} + \frac{1-\beta}{\beta} = \frac{3}{4},$$

$$\frac{3+\beta}{-2-\beta} + \frac{1-\beta}{\beta} = \frac{3}{4},$$

$$-\frac{3+\beta}{\beta+2} + \frac{1-\beta}{\beta} = \frac{3}{4},$$

$$-\left(1 + \frac{1}{\beta+2}\right) + \frac{1}{\beta} - 1 = \frac{3}{4},$$

$$-\frac{1}{\beta+2} + \frac{1}{\beta} = \frac{3}{4} + 2,$$

$$-\frac{1}{\beta+2} + \frac{1}{\beta} = \frac{11}{4},$$

$$-4\beta + 4(\beta + 2) = 11\beta(\beta + 2),$$

$$11\beta^2 + 22\beta - 8 = 0,$$

$$11(\beta + 1)^2 - 8 - 11 = 0,$$

hence

$$\beta = -1 \pm \sqrt{\frac{19}{11}}.$$

If $\beta = -1 - \sqrt{\frac{19}{11}}$, then

$$\alpha = -2 - \beta = -2 - \left(-1 - \sqrt{\frac{19}{11}}\right) = -1 + \sqrt{\frac{19}{11}},$$

using the second equation of (1)

$$p = \alpha\beta = -\left(-1 + \sqrt{\frac{19}{11}}\right)\left(1 + \sqrt{\frac{19}{11}}\right) = -\left(\frac{19}{11} - 1\right) = -\frac{8}{11}.$$

If $\beta = -1 + \sqrt{\frac{19}{11}}$, then

$$\alpha = -2 - \beta = -2 - \left(-1 + \sqrt{\frac{19}{11}}\right) = -1 - \sqrt{\frac{19}{11}},$$

using the second equation of (1)

$$p = \alpha\beta = -\left(1 + \sqrt{\frac{19}{11}}\right)\left(-1 + \sqrt{\frac{19}{11}}\right) = -\left(\frac{19}{11} - 1\right) = -\frac{8}{11}.$$

In both cases

$$\alpha\beta = p = -\frac{8}{11} \quad \textbf{(6)}$$

Substituting $\alpha + \beta = -2$ from (1), $\alpha\beta = -\frac{8}{11}$ from (6) into (3) we get

$$q = 4\frac{1 - (-2) + \left(-\frac{8}{11}\right)}{\left(-\frac{8}{11}\right)} = 4 \cdot \frac{3 - \frac{8}{11}}{-\frac{8}{11}} = 4 \cdot \frac{\frac{3 \cdot 11 - 8}{11}}{-\frac{8}{11}} = 4 \cdot \frac{\frac{25}{11}}{-\frac{8}{11}} = -\frac{25}{2} =$$

$= -12.5.$

Answer: $p = -\frac{8}{11}$; $q = -12.5$.