Answer on Question #63548 – Math – Statistics and Probability

The exponential distribution with rate parameter $\mu > 0$ is a continuous distribution on $[0, \infty)$ with density

$f(t) = \mu \exp(-\mu t), t > 0$

Question

1. Compute the cumulative distribution function defined by
$F(t) := P(X \in [0, t])$.

Solution

The cdf is

$$F(t) = P(X \le t) = \int_{-\infty}^{t} f(t)\,dt = \begin{cases} 1 - e^{(-\mu t)}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

Answer: $F(t) = \begin{cases} 1 - e^{(-\mu t)}, & t \ge 0 \\ 0, & t < 0 \end{cases}$

Question

2. Compute $P(s < X \le t)$.

Solution

$$P(s < X \le t) = P(X \le t) - P(X \le s) = 1 - e^{-\mu t} - (1 - e^{-\mu s}) = e^{-\mu s} - e^{-\mu t}.$$

$$P(s < X \le t) = \int_{s}^{t} f(t)\,dt = \int_{s}^{t} \mu e^{(-\mu t)}\,dt = -e^{(-\mu t)} \Big|_{s}^{t} = e^{-\mu s} - e^{-\mu t}.$$

Answer: $P(s < X \le t) = e^{-\mu s} - e^{-\mu t}$.

Question

3. Find $P(X \in [1, 2] \cup [3, 4])$.

Solution

$$P(X \in [1,2] \cup [3,4]) = P(X \le 2) - P(X \le 1) + P(X \le 4) - P(X \le 3) =$$

$$= 1 - e^{-2\mu} - (1 - e^{-\mu}) + 1 - e^{-4\mu} - (1 - e^{-3\mu}) = e^{-\mu} + e^{-3\mu} - e^{-2\mu} - e^{-4\mu} =$$

$$= \frac{e^{3\mu} - e^{2\mu} + e^{\mu} - 1}{e^{4\mu}} = \frac{(e^{\mu} - 1)(e^{2\mu} + 1)}{e^{4\mu}}$$

Answer: $\frac{(e^{\mu}-1)(e^{2\mu}+1)}{e^{4\mu}}$.

Question

4. Compute the conditional probability P (X ∈ [3, 4] | X ∈ [1, 4]) .

Solution

$$P(X \in [3,4]|X \in [1,4]) = \frac{P(X \in [3,4] \text{ and } X \in [1,4])}{P(X \in [1,4])} = \frac{P(X \in [3,4])}{P(X \in [1,4])} =$$

$$= \frac{P(X \le 4) - P(X \le 3)}{P(X \le 4) - P(X \le 1)} = \frac{e^{-3\mu} - e^{-4\mu}}{e^{-\mu} - e^{-4\mu}} = \frac{e^{\mu} - 1}{e^{3\mu} - 1} = \frac{1}{e^{2\mu} + e^{\mu} + 1}.$$

Answer: $\frac{1}{e^{2\mu}+e^{\mu}+1}$.

Question

5. Compute the conditional probability P (X > t + s | X > s) for s, t ≥ 0.

Solution

$$P(X > t+s|X > s) = \frac{P(X > s+t \text{ and } X > s)}{P(X > s)} = \frac{P(X > s+t\,)}{P(X > s)} = \frac{e^{-\mu(t+s)}}{e^{-\mu s}} =$$

$$= e^{-\mu t}.$$

Answer: $P(X > t+s|X > s) = e^{-\mu t}$.

Question

6. Find the mean of the exponential distribution with rate parameter μ > 0.

Solution

The mean is

$$EX = \int_0^{\infty} tf(t)\,dt = \int_0^{\infty} t\mu e^{(-\mu t)}\,dt = -te^{(-\mu t)}\Big|_0^{\infty} + \int_0^{\infty} e^{(-\mu t)}\,dt =$$

$$= 0 - 0 - \frac{1}{\mu}e^{(-\mu t)}\Big|_0^{\infty} = 0 + \frac{1}{\mu} = \frac{1}{\mu}.$$

Answer: $EX = \frac{1}{\mu}$.