Answer on Question #63508 – Math – Calculus

Question

1. If
   ϕ=2xz4−x2y,
   find
   |∇ϕ|

Solution

$$\varphi = 2xz^4 - x^2y,$$

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j} + \frac{\partial\varphi}{\partial z}\mathbf{k} = (2z^4 - 2xy)\mathbf{i} - x^2\mathbf{j} + 8xz^3\mathbf{k}$$

$$|\nabla\varphi| = \sqrt{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2} = \sqrt{(2z^4 - 2xy)^2 + x^4 + 64x^2z^6} =$$

$$= \sqrt{(2z^4 - 2xy)^2 + x^2(x^2 + 64z^6)}$$

ANSWER: $|\nabla\varphi| = \sqrt{(2z^4 - 2xy)^2 + x^2(x^2 + 64z^6)}$**.**

Question

2. If
ϕ (x,y,z)=3x2y−y3z2,
find
∇ϕ
at point (1,-2,-1)

Solution

$\varphi = 3x^2y - y^3z^2$

$\nabla\varphi = \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j} + \frac{\partial\varphi}{\partial z}\mathbf{k} = 6xy\mathbf{i} + (3x^2 - 3y^2z^2)\mathbf{j} - 2y^3z\mathbf{k}.$

At point (1,-2,-1)

$$\nabla\varphi|_{(1,-2,-1)} = 6 \cdot 1 \cdot (-2)\mathbf{i} + (3 \cdot 1 - 3 \cdot 4 \cdot 1)\mathbf{j} - 2 \cdot (-8) \cdot (-1)\mathbf{k} = -12\mathbf{i} - 9\mathbf{j} - 16\mathbf{k}$$

ANSWER:

$\nabla\varphi|_{(1,-2,-1)} = -12\mathbf{i} - 9\mathbf{j} - 16\mathbf{k}$

Question

3. Find a unit normal to the surface
x2y+2xz=4
at point (2,-2,3)

Solution

Let $\varphi(x, y, z) = x^2y + 2xz - 4 = 0.$

Then the gradient gives a normal vector:

$\mathbf{n} = \nabla\varphi = \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j} + \frac{\partial\varphi}{\partial z}\mathbf{k} = (2xy + 2z)\mathbf{i} + x^2\mathbf{j} + 2x\mathbf{k}.$

At point (2,-2,3) $\mathbf{n} = -2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$.

Find a magnitude of $\mathbf{n}$ there:

$$\|\mathbf{n}\| = \sqrt{4+16+16} = \sqrt{36} = 6$$

Then a unit normal to the surface at point (2,-2,3) is

$\mathbf{n}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} = \left(-\frac{2}{6}, \frac{4}{6}, \frac{4}{6}\right) = \left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ or $\mathbf{n}_2 = -\frac{\mathbf{n}}{\|\mathbf{n}\|} = -\left(-\frac{2}{6}, \frac{4}{6}, \frac{4}{6}\right) = \left(\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$.

ANSWER: $\left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ or $\left(\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$.

Question

4. Let

ϕ (x,y,z)=xy2z

and

A=xzi−xy2j+yz2k,

find

∂3∂x2∂z(ϕA)

Solution

$\varphi(x,y,z) = xy^2z$ $\mathrm{A} = xz\mathrm{i} - xy^2\mathrm{j} + yz^2\mathrm{k}$

$\varphi\mathrm{A} = x^2y^2z^2\mathrm{i} - x^2y^4z\mathrm{j} + xy^3z^3\mathrm{k}$

$\frac{\partial^3}{\partial x^2\, \partial z}(\varphi\mathrm{A}) = \frac{\partial^3}{\partial x^2\, \partial z}\left(x^2y^2z^2\mathrm{i} - x^2y^4z\mathrm{j} + xy^3z^3\mathrm{k}\right) = 4y^2z\mathrm{i} - 2y^4\mathrm{j} + 0\mathrm{k}$

ANSWER: $\frac{\partial^3}{\partial x^2\, \partial z}(\varphi\mathrm{A}) = 4y^2z\mathrm{i} - 2y^4\mathrm{j}$

Question

5. Given that

ϕ=2x2y−xz3

find

∇2ϕ

Solution

$\varphi = 2x^2y - xz^3$

$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 4y - 6xz$

ANSWER: $\nabla^2\varphi = 4y - 6xz$

Question

6. If

A=xz3i−2x2yzj+2yz4,

find

∇×A

at point (1,-1,1).

Solution

$\mathbf{A} = xz^3\mathbf{i} - 2x^2yz\mathbf{j} + 2yz^4\mathbf{k}$

$$\nabla\times A = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix} = (2z^4 + 2x^2y)\mathbf{i} - (0 - 3xz^2)\mathbf{j} + (-4xyz - 0)\mathbf{k} =$$

$= 2(z^4 + x^2y)\mathbf{i} + 3xz^2\mathbf{j} - 4xyz\mathbf{k}$

at point (1,-1,1) we get $\nabla\times A|_{(1,-1,1)} = 0\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} = 3\mathbf{j} + 4\mathbf{k}$.

ANSWER: $\nabla\times A|_{(1,-1,1)} = 3\mathbf{j} + 4\mathbf{k}$.

Question

7. Given that

A=A1i+A2j+A3k

and

r=xi+yj+zk,

evaluate

∇ · (A×r)

if

∇×A=0

Solution

Use the formula $\nabla\cdot(\mathbf{A}\times\mathbf{r}) = \mathbf{r}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}(\nabla\times\mathbf{r})$

Since $\nabla\times A = 0$ then $\nabla\cdot(\mathbf{A}\times\mathbf{r}) = -\mathbf{A}(\nabla\times\mathbf{r})$

But $\nabla\times \mathrm{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$

Then $\nabla\cdot(\mathbf{A}\times\mathbf{r}) = 0$

ANSWER: $\nabla\cdot(\mathbf{A}\times\mathbf{r}) = 0$

Question

8. Let A=x2yi−2xzj+2yzk

A=x2yi−2xzj+2yzk,

find Curl curl A.

Solution

$\mathbf{A} = x^2y\mathbf{i} - 2xz\mathbf{j} + 2yz\mathbf{k}$

$$\mathrm{curl}\,\mathbf{A} = \nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix} = (2z + 2x)\mathbf{i} - (0 - 0)\mathbf{j} + (-2z - x^2)\mathbf{k} =$$

$= 2(x + z)\mathbf{i} - (x^2 + 2z)\mathbf{k}$

$$\text{curl curl } \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2(\mathrm{x}+\mathrm{z}) & 0 & -(x^2+2z) \end{vmatrix} = (0-0)\mathbf{i} - (-2\mathrm{x}-2)\mathbf{j} + (0-0)\mathbf{k} =$$

$= 2(1+\mathrm{x})\mathbf{j}$

ANSWER: curl curl $\mathbf{A} = 2(1+\mathrm{x})\mathbf{j}$

Question

9. Given A=2x2i−3yzj+xz2k

A=2x2i−3yzj+xz2k

and

ϕ=2z−x3y,

find

A⋅∇ϕ

at point (1,-1,1)

Solution

$\mathbf{A} = 2x^2\mathbf{i} - 3yz\mathbf{j} + xz^2\mathbf{k} \quad \phi = 2z - x^3y$

$\nabla\varphi = \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j} + \frac{\partial\varphi}{\partial z}\mathbf{k} = -3x^2y\mathbf{i} - x^3\mathbf{j} + 2\mathbf{k}$

$\boldsymbol{A} \cdot \nabla\varphi == (2x^2\mathbf{i} - 3yz\mathbf{j} + xz^2\mathbf{k}) \cdot (-3x^2y\mathbf{i} - x^3\mathbf{j} + 2\mathbf{k}) = -6x^4y + 3x^3yz + 2xz^2$

at point (1,-1,1) we get

$\boldsymbol{A} \cdot \nabla\varphi|_{(1,-1,1)} = -6 \cdot 1^4(-1) + 3 \cdot 1^3(-1) \cdot 1 + 2 \cdot 1 \cdot 1^2 = 6 - 3 + 2 = 5$

ANSWER: $A \cdot \nabla\varphi|_{(1,-1,1)} = 5$

Question

10. Find the directional derivative of

ϕ=x2yz+4xz2

at (1,-2,-1) in the direction

2i−j−2k

at point (1,-1,1)

Solution

$\varphi = x^2yz + 4xz^2,\ \mathrm{a} = 2\mathrm{i} - \mathrm{j} - 2\mathrm{k}$

$\frac{\partial\varphi}{\partial a} = \frac{\partial\varphi}{\partial x}\cos\alpha + \frac{\partial\varphi}{\partial y}\cos\beta + \frac{\partial\varphi}{\partial z}\cos\gamma$

$\cos\alpha = \frac{2}{\sqrt{4+1+4}} = \frac{2}{3}, \quad \cos\beta = \frac{-1}{\sqrt{4+1+4}} = -\frac{1}{3}, \quad \cos\gamma = \frac{-2}{\sqrt{4+1+4}} = \frac{-2}{3}$

$\frac{\partial\varphi}{\partial a} = (2xyz + 4z^2)\frac{2}{3} - x^2z\frac{1}{3} + (x^2y + 8xz)\frac{2}{3}.$

At point (1,-1,1) we get the directional derivative:

$\left.\frac{\partial\varphi}{\partial a}\right|_{(1,-1,1)} = (2xyz + 4z^2)\frac{2}{3} - x^2z\frac{1}{3} + (x^2y + 8xz)\frac{2}{3} = \frac{4}{3} - \frac{1}{3} + \frac{14}{3} = \frac{17}{3}$

ANSWER: $\left.\frac{\partial\varphi}{\partial a}\right|_{(1,-1,1)} = \frac{17}{3}.$