ANSWER ON QUESTION #62927 – MATH - CALCULUS

QUESTION

Find limit $\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$, where F_n is a Fibonacci number, e is Euler's number.

SOLUTION

We know that

$$F_n = F_{n-1} + F_{n-2}. \quad \textbf{(1)}$$

$$F_1 = F_2 = 1, F_3 = F_1 + F_2 = 1 + 1 = 2, F_4 = F_2 + F_3 = 1 + 2 = 3, F_5 = F_3 + F_4 = 2 + 3 = 5, \ldots$$

Use (1) and let

$$L = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{F_{n-1} + F_n}{F_n} = \lim_{n\to\infty} \frac{F_{n-1}}{F_n} + 1 = 1 + \frac{1}{\lim_{n\to\infty} \frac{F_n}{F_{n-1}}} \ \textbf{(2).}$$

On the other hand, $n \to \infty$, $m = n - 1 \to \infty$ and

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = |n - 1 = m, \ n = m + 1| = \lim_{m\to\infty} \frac{F_{m+1}}{F_m} = L \ \textbf{(3)}$$

It follows from (3) that

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = L. \ \textbf{(4)}$$

Substituting (4) into (2)

$$L = 1 + \frac{1}{L}.$$

Multiplying both sides by L

$$L^2 = L + 1.$$

Hence

$$L^2 - L - 1 = 0.$$

Solutions of this quadratic equation are

$$L_1 = \frac{1 - \sqrt{(-1)^2 - 4\cdot(-1)}}{2} = \frac{1 - \sqrt{5}}{2}, L_2 = \frac{1 + \sqrt{(-1)^2 - 4\cdot(-1)}}{2} = \frac{1 + \sqrt{5}}{2}.$$

Because all terms F_n, F_{n-1} are greater than 1, so $\lim_{n\to\infty} \frac{F_n}{F_{n-1}}$ can't be negative respectively.

Thus, the correct answer will be

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = L = \frac{1 + \sqrt{5}}{2} \approx 1.618. \text{ Hence } L \neq e, \text{ where } e \approx 2.718 \text{ is Euler's number.}$$

ANSWER: $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1 + \sqrt{5}}{2}$.