Question

Prove that

$$\frac{sin(A+B)+cos(A-B)}{sin(A-B)+cos(A+B)} = tan\left(\frac{\pi}{4}+B\right). \textbf{ (1)}$$

Solution

Recall that

$$\sin(A+B) = \sin A \cos B + \cos A \sin B, \qquad \sin(A-B) = \sin A \cos B - \cos A \sin B,$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B, \qquad \cos(A-B) = \cos A \cos B + \sin A \sin B.$$

Transforming the left-hand side of (1) get

$$\frac{sin(A+B)+cos(A-B)}{sin(A-B)+cos(A+B)} = \frac{(sinA\,cosB+cosA\,sinB)+(cosA\,cosB+\sin A\,sinB)}{(sinA\,cosB-cosAsinB)+(cosA\,cosB-sinAsinB)}$$

$$= \frac{sinA\,cosB+cosA\,sinB+cosA\,cosB+\sin A\,sinB}{sinA\,cosB-cosAsinB+cosA\,cosB-sinAsinB}$$

$$= \frac{(sinA\,cosB+cosA\,cosB)+(cosA\,sinB+\sin A\,sinB)}{(sinA\,cosB+cosA\,cosB)-(cosAsinB+sinAsinB)} =$$

$$= \frac{\cos B(sinA+cosA)+\sin B(cosA+\sin A)}{\cos B(sinA+cosA)-sinB(cosA+sinA)} = \frac{(\cos B+\sin B)(\cos A+\sin A)}{(\cos A+\sin A)(\cos B-\sin B)} = \frac{\cos B+\sin B}{\cos B-\sin B}.$$

We proved that

$$\frac{sin(A+B)+cos(A-B)}{sin(A-B)+cos(A+B)} = \frac{\cos B+\sin B}{\cos B-\sin B}. \textbf{ (2)}$$

Transforming the right-hand side of (1) get

$$\tan\left(\frac{\pi}{4}+B\right) = \frac{\tan\left(\frac{\pi}{4}\right)+\tan B}{1-\tan\left(\frac{\pi}{4}\right)\tan B} = \frac{1+\tan B}{1-\tan B} = \frac{1+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin B}{\cos B}} = \frac{\dfrac{\cos B+\sin B}{\cos B}}{\dfrac{\cos B-\sin B}{\cos B}} =$$

$$= \frac{\cos B+\sin B}{\cos B(\cos B-\sin B)}\cdot\cos B = \frac{\cos B+\sin B}{\cos B-\sin B}.$$

We proved that

$$\tan\left(\frac{\pi}{4}+B\right) = \frac{\cos B+\sin B}{\cos B-\sin B}. \quad\textbf{(3)}$$

It follows from (2) and (3) that

$$\frac{sin(A+B)+cos(A-B)}{sin(A-B)+cos(A+B)} = \frac{\cos B+\sin B}{\cos B-\sin B} = \tan\left(\frac{\pi}{4}+B\right).$$

Hence

$$\frac{sin(A+B)+cos(A-B)}{sin(A-B)+cos(A+B)} = \tan\left(\frac{\pi}{4}+B\right).$$

Q.E.D.