Answer on Question #62872 – Math – Calculus

Question

evaluate the following using Beta function:
I=integration(dx/sqrt(1-(x)^n)) limits: from zero to one

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^n}}.$$

Solution

The Beta function can be defined as

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt.$$

Then our function can be transformed in the following way:

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^n}} = \int_0^1 (1-x^n)^{-\frac{1}{2}}dx = \begin{bmatrix} x^n = t & \\ x = t^{\frac{1}{n}} & x=0, t=0 \\ dx = \frac{1}{n}t^{\frac{1}{n}-1}dt & x=1, t=1 \end{bmatrix} = \int_0^1 (1-t)^{-\frac{1}{2}} \cdot \frac{1}{n} t^{\frac{1}{n}-1}dt =$$

$$= \frac{1}{n}\int_0^1 t^{\frac{1}{n}-1}(1-t)^{-\frac{1}{2}}dt = \frac{1}{n}\int_0^1 t^{\frac{1}{n}-1}(1-t)^{\frac{1}{2}-1}dt = \frac{1}{n}B\left(\frac{1}{n},\frac{1}{2}\right).$$

So, $a = \frac{1}{n}, b = \frac{1}{2}$. Let's use the following formula to turn the Beta function into the Gamma function:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Therefore

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{1}{n}B\left(\frac{1}{n},\frac{1}{2}\right) = \frac{1}{n}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{n}\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}.$$

Answer: $I = \frac{1}{n}B\left(\frac{1}{n},\frac{1}{2}\right) = \frac{\sqrt{\pi}}{n}\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$.