Answer on Question #62837 – Math – Calculus

Question

evaluate the following using Beta function:
I=integration(dx/sqrt(1-(x)^n)) limits: from zero to one

Solution

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^n}}$$

Substituting $z = x^n$ obtain $x = \sqrt[n]{z} = z^{\frac{1}{n}}$ and $dx = \left(z^{\frac{1}{n}}\right)' dz = \frac{1}{n} z^{\frac{1}{n}-1} dz$.

If $x = 0$, then $z = 0$; if $x = 1$, then $z = 1$. In other words, limits of integration do not change.

After this step we proceed with the calculation of integral:

$$I = \frac{1}{n}\int_0^1 z^{\frac{1}{n}-1} \frac{1}{\sqrt{1-z}} dz = \frac{1}{n}\int_0^1 z^{\frac{1}{n}-1} (1-z)^{-\frac{1}{2}} dz = \frac{1}{n}\int_0^1 z^{\frac{1}{n}-1} (1-z)^{\frac{1}{2}-1} dz.$$

By definition, the Beta function is

$$B(p,q) = \int_0^1 z^{p-1} (1-z)^{q-1} dz \qquad (p > 0, q > 0).$$

In our case $p = \frac{1}{n}$ and $q = \frac{1}{2}$.

Then

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{1}{n}\int_0^1 z^{\frac{1}{n}-1} (1-z)^{\frac{1}{2}-1} dz = \frac{1}{n} B\left(\frac{1}{n}, \frac{1}{2}\right).$$

Answer: $I = \int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{1}{n} B\left(\frac{1}{n}, \frac{1}{2}\right).$