Answer on Question #62622 – Math – Linear Algebra

D. Solving a System of Linear Equation.

Direction: Use an inverse matrix to solve each system of linear equation.

Question

1. (a)

$$\begin{cases} x + 2y = -1; \\ x - 2y = 3. \end{cases}$$

Solution

$A = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}; X = \begin{bmatrix} x \\ y \end{bmatrix}; B = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$

The system

$$AX = B \ \textbf{(1)}$$

is given.

Using the inverse A^{-1} of the matrix A find

$$X = A^{-1}B.$$

Next,

$\det A = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = 1 \cdot (-2) - 1 \cdot 2 = -4 \neq 0 \Rightarrow A^{-1}$ exists.

Find the minors M_{ij} and the cofactors C_{ij} of the matrix A:

$M_{11} = -2, M_{12} = 1, M_{21} = 2, M_{22} = 1, C_{11} = (-1)^{1+1}M_{11} = M_{11} = -2,$
$C_{12} = (-1)^{1+2}M_{12} = -M_{12} = -1, C_{21} = (-1)^{2+1}M_{21} = -M_{21} = -2,$
$C_{22} = (-1)^{2+2}M_{22} = M_{22} = 1.$

Write the matrix of cofactors:

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix}.$$

The transposed matrix of cofactors:

$$C^T = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -1 & 1 \end{bmatrix}.$$

Find the inverse matrix:

$$A^{-1} = \frac{1}{\det A} C^T = -\frac{1}{4} \cdot \begin{bmatrix} -2 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-2}{-4} & \frac{-2}{-4} \\ \frac{-1}{-4} & \frac{1}{-4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

Then the solution of the system (1) is

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 3 \\ \frac{1}{4} \cdot (-1) - \frac{1}{4} \cdot 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{3}{2} \\ -\frac{1}{4} - \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Answer: $x = 1, y = -1.$

Question

1. (b)

$$\begin{cases} x + 2y = 10; \\ x - 2y = -6. \end{cases}$$

Solution

$A = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}; X = \begin{bmatrix} x \\ y \end{bmatrix}; B = \begin{bmatrix} 10 \\ -6 \end{bmatrix}.$

The system

$$AX = B \ \mathbf{(2)}$$

is given.

Using the inverse A^{-1} of the matrix A find

$$X = A^{-1}B.$$

Next,

$$\det A = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = 1 \cdot (-2) - 1 \cdot 2 = -4 \neq 0 => A^{-1} \text{ exists.}$$

Find the minors M_{ij} and the cofactors C_{ij} of the matrix A:

$M_{11} = -2$, $M_{12} = 1$, $M_{21} = 2$, $M_{22} = 1$, $C_{11} = (-1)^{1+1}M_{11} = M_{11} = -2$,
$C_{12} = (-1)^{1+2}M_{12} = -M_{12} = -1$, $C_{21} = (-1)^{2+1}M_{21} = -M_{21} = -2$,
$C_{22} = (-1)^{2+2}M_{22} = M_{22} = 1$.

Write the matrix of cofactors:

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix}.$$

The transposed matrix of cofactors:

$$C^T = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -1 & 1 \end{bmatrix}.$$

Find the inverse matrix:

$$A^{-1} = \frac{1}{\det A} \cdot C^T = -\frac{1}{4}\begin{bmatrix} -2 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-2}{-4} & \frac{-2}{-4} \\ \frac{-1}{-4} & \frac{1}{-4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

Then the solution of the system (2) is

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \cdot B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 10 \\ -6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \cdot 10 + \frac{1}{2} \cdot (-6) \\ \frac{1}{4} \cdot 10 - \frac{1}{4} \cdot (-6) \end{bmatrix} = \begin{bmatrix} \frac{10}{2} - \frac{6}{2} \\ \frac{10}{4} + \frac{6}{4} \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Answer: $x = 2$, $y = 4$.

Question

2. (a)

$$\begin{cases} 2x - y = -3; \\ 2x + y = 7. \end{cases}$$

Solution

$A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \end{bmatrix}; B = \begin{bmatrix} -3 \\ 7 \end{bmatrix}.$

The system

$$AX = B \ \textbf{(3)}$$

is given.
Using the inverse A^{-1} of the matrix A find

$$X = A^{-1}B.$$

Next,

$$\det A = \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = 2 \cdot 1 - (-1) \cdot 2 = 4 \neq 0 => A^{-1}\text{exists.}$$

Find the minors M_{ij} and the cofactors C_{ij} of the matrix A:
$M_{11} = 1, M_{12} = 2, M_{21} = -1, M_{22} = 2, C_{11} = (-1)^{1+1}M_{11} = M_{11} = 1,$
$C_{12} = (-1)^{1+2}M_{12} = -M_{12} = -2, C_{21} = (-1)^{2+1}M_{21} = -M_{21} = 1,$
$C_{22} = (-1)^{2+2}M_{22} = M_{22} = 2.$
Write the matrix of cofactors:

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}.$$

The transposed matrix of cofactors:

$$C^T = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}.$$

Find the inverse matrix:

$$A^{-1} = \frac{1}{\det A} C^T = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{-2}{4} & \frac{2}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Then the solution of the system (3) is

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \cdot (-3) + \frac{1}{4} \cdot 7 \\ -\frac{1}{2} \cdot (-3) + \frac{1}{2} \cdot 7 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} + \frac{7}{4} \\ \frac{3}{2} + \frac{7}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Answer: $x = 1, y = 5.$

Question

2. (b)

$$\begin{cases} 2x - y = -1; \\ 2x + y = -3. \end{cases}$$

Solution

$A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \end{bmatrix}; B = \begin{bmatrix} -1 \\ -3 \end{bmatrix}.$

The system

$$AX = B \ \textbf{(4)}$$

is given.
Using the inverse A^{-1} of the matrix A find

$$X = A^{-1}B.$$

Next,

$$\det A = \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = 2 \cdot 1 - (-1) \cdot 2 = 4 \neq 0 => A^{-1}\text{exists.}$$

Find the minors M_{ij} and the cofactors C_{ij} of the matrix A:

$M_{11} = 1, M_{12} = 2, M_{21} = -1, M_{22} = 2, C_{11} = (-1)^{1+1}M_{11} = M_{11} = 1,$
$C_{12} = (-1)^{1+2}M_{12} = -M_{12} = -2, C_{21} = (-1)^{2+1}M_{21} = -M_{21} = 1,$
$C_{22} = (-1)^{2+2}M_{22} = M_{22} = 2.$
Write the matrix of cofactors:

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}.$$

The transposed matrix of cofactors:

$$C^T = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}.$$

Find the inverse matrix:

$$A^{-1} = \frac{1}{\det A} C^T = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{-2}{4} & \frac{2}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Then the solution of the system (4) is

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \cdot (-1) + \frac{1}{4} \cdot (-3) \\ -\frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot (-3) \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} - \frac{3}{4} \\ \frac{1}{2} - \frac{3}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Answer: $x = -1, y = -1.$

Question

3. (a)

$$\begin{cases} x + 2y + z = 2; \\ x + 2y - z = 4; \\ x - 2y + z = -2. \end{cases}$$

Solution

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & -1 \\ 1 & -2 & 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}.$$

The system

$$AX = B \ \textbf{(5)}$$

is given.
Using the inverse A^{-1} of the matrix A find

$$X = A^{-1}B.$$

Next,

$$\det A = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & -1 \\ 1 & -2 & 1 \end{vmatrix} = |expand\ along\ the\ first\ row\ |$$

$$= 1 \cdot \begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} =$$

$$= \big(2 \cdot 1 - (-2) \cdot (-1)\big) - 2 \cdot \big(1 \cdot 1 - 1 \cdot (-1)\big) + (1 \cdot (-2) - 1 \cdot 2) = 0 - 4 - 4 =$$

$$= -8 \neq 0 => A^{-1} \text{ exists.}$$

Find the minors M_{ij} and the cofactors C_{ij} of the matrix A:

$M_{11} = \begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} = 2 \cdot 1 - (-2) \cdot (-1) = 0,$
$M_{12} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 \cdot 1 - 1 \cdot (-1) = 2,$
$M_{13} = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = 1 \cdot (-2) - 1 \cdot 2 = -4,$
$M_{21} = \begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} = 2 \cdot 1 - (-2) \cdot 1 = 4,$
$M_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 \cdot 1 - 1 \cdot 1 = 0,$
$M_{23} = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = 1 \cdot (-2) - 1 \cdot 2 = -4,$
$M_{31} = \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} = 2 \cdot (-1) - 2 \cdot 1 = -4,$
$M_{32} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 1 \cdot (-1) - 1 \cdot 1 = -2,$
$M_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 1 \cdot 2 - 1 \cdot 2 = 0,$
$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 0,\ C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -2,$
$C_{13} = (-1)^{1+3} M_{13} = M_{13} = -4,\ C_{21} = (-1)^{2+1} M_{21} = -M_{21} = -4,$
$C_{22} = (-1)^{2+2} M_{22} = M_{22} = 0,\ C_{23} = (-1)^{2+3} M_{23} = -M_{23} = 4,$
$C_{31} = (-1)^{3+1} M_{31} = M_{31} = -4,\ C_{32} = (-1)^{3+2} M_{32} = -M_{32} = 2,$
$C_{33} = (-1)^{3+3} M_{33} = M_{33} = 0.$
Write the matrix of cofactors:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 0 & -2 & -4 \\ -4 & 0 & 4 \\ -4 & 2 & 0 \end{bmatrix}.$$

The transposed matrix of cofactors:

$$C^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 0 & -4 & -4 \\ -2 & 0 & 2 \\ -4 & 4 & 0 \end{bmatrix}.$$

Find the inverse matrix:

$$A^{-1} = \frac{1}{\det A} C^T = -\frac{1}{8} \begin{bmatrix} 0 & -4 & -4 \\ -2 & 0 & 2 \\ -4 & 4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{0}{-8} & \frac{-4}{-8} & \frac{-4}{-8} \\ \frac{-2}{-8} & \frac{0}{-8} & \frac{2}{-8} \\ \frac{-4}{-8} & \frac{4}{-8} & \frac{0}{-8} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}.$$

Then the solution of the system (5) is

$$X=\begin{bmatrix}x\\y\\z\end{bmatrix}=A^{-1}B=\begin{bmatrix}0&\frac{1}{2}&\frac{1}{2}\\\frac{1}{4}&0&-\frac{1}{4}\\\frac{1}{2}&-\frac{1}{2}&0\end{bmatrix}\cdot\begin{bmatrix}2\\4\\-2\end{bmatrix}=\begin{bmatrix}0\cdot2+\frac{1}{2}\cdot4+\frac{1}{2}\cdot(-2)\\\frac{1}{4}\cdot2+0\cdot4-\frac{1}{4}\cdot(-2)\\\frac{1}{2}\cdot2-\frac{1}{2}\cdot4+0\cdot(-2)\end{bmatrix}=$$

$$=\begin{bmatrix}0+2-1\\\frac{1}{2}+0+\frac{1}{2}\\1-2-0\end{bmatrix}=\begin{bmatrix}1\\1\\-1\end{bmatrix}$$

Answer: $x=1$, $y=1$, $z=-1$.

Question

3. (b)

$$\begin{cases}x+2y+z=1;\\x+2y-z=3;\\x-2y+z=-3.\end{cases}$$

Solution

$$A=\begin{bmatrix}1&2&1\\1&2&-1\\1&-2&1\end{bmatrix};X=\begin{bmatrix}x\\y\\z\end{bmatrix};B=\begin{bmatrix}1\\3\\-3\end{bmatrix}.$$

The system

$$AX=B \ \textbf{(6)}$$

is given.

Using the inverse A^{-1} of the matrix A find

$$X=A^{-1}B.$$

Next,

$$\det A=\begin{vmatrix}1&2&1\\1&2&-1\\1&-2&1\end{vmatrix}=|expand\ along\ the\ first\ row|$$

$$=1\cdot\begin{vmatrix}2&-1\\-2&1\end{vmatrix}-2\cdot\begin{vmatrix}1&-1\\1&1\end{vmatrix}+1\cdot\begin{vmatrix}1&2\\1&-2\end{vmatrix}=$$

$=0-4-4=-8\neq0 \Rightarrow A^{-1}$ exists.

Find the minors M_{ij} and the cofactors C_{ij} of the matrix A:

$M_{11}=\begin{vmatrix}2&-1\\-2&1\end{vmatrix}=2\cdot1-(-2)\cdot(-1)=0,$

$M_{12}=\begin{vmatrix}1&-1\\1&1\end{vmatrix}=1\cdot1-1\cdot(-1)=2,$

$M_{13}=\begin{vmatrix}1&2\\1&-2\end{vmatrix}=1\cdot(-2)-1\cdot2=-4,$

$M_{21}=\begin{vmatrix}2&1\\-2&1\end{vmatrix}=2\cdot1-(-2)\cdot1=4,$

$M_{22}=\begin{vmatrix}1&1\\1&1\end{vmatrix}=1\cdot1-1\cdot1=0,$

$M_{23}=\begin{vmatrix}1&2\\1&-2\end{vmatrix}=1\cdot(-2)-1\cdot2=-4,$

$M_{31}=\begin{vmatrix}2&1\\2&-1\end{vmatrix}=2\cdot(-1)-2\cdot1=-4,$

$M_{32} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 1 \cdot (-1) - 1 \cdot 1 = -2,$
$M_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 1 \cdot 2 - 1 \cdot 2 = 0,$
$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 0,\ C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -2,$
$C_{13} = (-1)^{1+3} M_{13} = M_{13} = -4,\ C_{21} = (-1)^{2+1} M_{21} = -M_{21} = -4,$
$C_{22} = (-1)^{2+2} M_{22} = M_{22} = 0,\ C_{23} = (-1)^{2+3} M_{23} = -M_{23} = 4,$
$C_{31} = (-1)^{3+1} M_{31} = M_{31} = -4,\ C_{32} = (-1)^{3+2} M_{32} = -M_{32} = 2,$
$C_{33} = (-1)^{3+3} M_{33} = M_{33} = 0.$
Write the matrix of cofactors:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 0 & -2 & -4 \\ -4 & 0 & 4 \\ -4 & 2 & 0 \end{bmatrix}.$$

The transposed matrix of cofactors:

$$C^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 0 & -4 & -4 \\ -2 & 0 & 2 \\ -4 & 4 & 0 \end{bmatrix}.$$

Find the inverse matrix:

$$A^{-1} = \frac{1}{\det A} C^T = -\frac{1}{8} \begin{bmatrix} 0 & -4 & -4 \\ -2 & 0 & 2 \\ -4 & 4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{0}{-8} & \frac{-4}{-8} & \frac{-4}{-8} \\ \frac{-2}{-8} & \frac{0}{-8} & \frac{2}{-8} \\ \frac{-4}{-8} & \frac{4}{-8} & \frac{0}{-8} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}.$$

Then the solution of the system (6) is

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot (-3) \\ \frac{1}{4} \cdot 1 + 0 \cdot 3 - \frac{1}{4} \cdot (-3) \\ \frac{1}{2} \cdot 1 - \frac{1}{2} \cdot 3 + 0 \cdot (-3) \end{bmatrix} =$$

$$= \begin{bmatrix} 0 + \frac{3}{2} - \frac{3}{2} \\ \frac{1}{4} + 0 + \frac{3}{4} \\ \frac{1}{2} - \frac{3}{2} - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Answer: $x = 0,\ y = 1,\ z = -1.$

E. Let A, B and C be:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix};\ B = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & -3 \end{bmatrix};\ C = \begin{bmatrix} 0 & 4 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}.$$

1. Find an elementary matrix E such that EA=B
2. Find an elementary matrix E such that EC=A
3. Find an elementary matrix E such that EB=A

Remark

Type I

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; E_1A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Type II

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}; E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}.$$

Type III

$$E_1 = \begin{bmatrix} 1 & 0 & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; E_1A = \begin{bmatrix} 1 & 0 & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11} + ma_{31} & a_{12} + ma_{32} & a_{13} + ma_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Question

1. Let A and B be

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}; \ B = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & -3 \end{bmatrix}.$$

Find an elementary matrix E such that $EA=B$

Solution

Type I: row1 and row3 of A are interchanged to get B.

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$EA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & -3 \end{bmatrix} = B.$$

Thus, $EA = B$.

Answer: $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Question

2. Let A and C be

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}; \ C = \begin{bmatrix} 0 & 4 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}.$$

Find an elementary matrix E such that $EC=A$

Solution

Type III: $m = -1$ (subtract the third row of C from the first row of C to get A).

$$E = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$EC = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 4 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix} = A.$$

Thus, $EC = A$.

Answer: $E = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Question

3. Let A and B be

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix};\ B = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & -3 \end{bmatrix}.$$

Find an elementary matrix E such that $EB=A$.

Solution

Type I: row1 and row3 of B are interchanged to get A.

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$EA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix} = A.$$

Thus, $EB = A$.

Answer: $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

F. Find the LU Factorization of the matrix.

Question

4.

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

Solution

Add twice $row1$ to $row2$ in the Gauss elimination method:

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \textbf{(*)}$$

According to (*), the inverse of $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$.

Let's multiply both sides of the equation (*) by this inverse (that is, by $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$):

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Simplify the previous formula and obtain the LU Factorization of the matrix $A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$:

$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = LU$, where $L = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Answer: $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Question

5.

$B = \begin{bmatrix} -2 & 1 \\ -6 & 4 \end{bmatrix}$

Solution

Add $(-3 \times row1)$ to $row2$ in the Gauss elimination method:

$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} -2 & 1 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} \quad \textbf{(**)}$$

Besides, the inverse of $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$.

Let's multiply both sides of the equation (**) by this inverse (that is, by $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$):

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} -2 & 1 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \times \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}$$

Simplify the previous formula and obtain the LU Factorization of the matrix $B = \begin{bmatrix} -2 & 1 \\ -6 & 4 \end{bmatrix}$:

$B = \begin{bmatrix} -2 & 1 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \times \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} = LU$, where $L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, $U = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}$.

Answer: $\begin{bmatrix} -2 & 1 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \times \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}$.

Question

6.

$C = \begin{bmatrix} 3 & 0 & 1 \\ 6 & 1 & 1 \\ -3 & 1 & 0 \end{bmatrix}$

Solution

Add $(-2 \times row1)$ to $row2$ in the Gauss elimination method:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 0 & 1 \\ 6 & 1 & 1 \\ -3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ -3 & 1 & 0 \end{bmatrix}$$

Add $(1 \times row1)$ to $row3$ in the Gauss elimination method:

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 1\end{bmatrix}\times\begin{bmatrix}3 & 0 & 1\\6 & 1 & 1\\-3 & 1 & 0\end{bmatrix}=\begin{bmatrix}3 & 0 & 1\\0 & 1 & -1\\0 & 1 & 1\end{bmatrix}$$

Add $(-1\times row2)$ to $row3$ in the Gauss elimination method:

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -1 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 1\end{bmatrix}\times\begin{bmatrix}3 & 0 & 1\\6 & 1 & 1\\-3 & 1 & 0\end{bmatrix}=\begin{bmatrix}3 & 0 & 1\\0 & 1 & -1\\0 & 0 & 2\end{bmatrix}\ \textbf{(***)}$$

The inverse of $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -1 & 1\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 1 & 1\end{bmatrix}$.

Let's multiply both sides of the equation (***) by this inverse (that is, by $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 1 & 1\end{bmatrix}$):

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 1 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -1 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 1\end{bmatrix}\times\begin{bmatrix}3 & 0 & 1\\6 & 1 & 1\\-3 & 1 & 0\end{bmatrix}=$$

$$=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 1 & 1\end{bmatrix}\times\begin{bmatrix}3 & 0 & 1\\0 & 1 & -1\\0 & 0 & 2\end{bmatrix}.$$

Simplify:

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 1\end{bmatrix}\times\begin{bmatrix}3 & 0 & 1\\6 & 1 & 1\\-3 & 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 1 & 1\end{bmatrix}\times\begin{bmatrix}3 & 0 & 1\\0 & 1 & -1\\0 & 0 & 2\end{bmatrix}\ \textbf{(****)}$$

The inverse of $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$.

Let's multiply both sides of the equation (****) by this inverse (that is, by $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$):

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-1 & 0 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 1\end{bmatrix}\times\begin{bmatrix}3 & 0 & 1\\6 & 1 & 1\\-3 & 1 & 0\end{bmatrix}=$$

$$=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-1 & 0 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 1 & 1\end{bmatrix}\times\begin{bmatrix}3 & 0 & 1\\0 & 1 & -1\\0 & 0 & 2\end{bmatrix}$$

Simplify:

$$\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 1\end{bmatrix}\times\begin{bmatrix}3 & 0 & 1\\6 & 1 & 1\\-3 & 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-1 & 1 & 1\end{bmatrix}\times\begin{bmatrix}3 & 0 & 1\\0 & 1 & -1\\0 & 0 & 2\end{bmatrix}\ \textbf{(*****)}$$

The inverse of $\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\0 & 0 & 1\end{bmatrix}$.

Let's multiply both sides of the equation (*****) by this inverse (that is, by $\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$):

$$\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\times\begin{bmatrix}3 & 0 & 1\\ 6 & 1 & 1\\ -3 & 1 & 0\end{bmatrix}=$$

$$=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & 1 & 1\end{bmatrix}\times\begin{bmatrix}3 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 2\end{bmatrix}$$

Simplify the previous formula and obtain the LU Factorization of the matrix $C=\begin{bmatrix}3 & 0 & 1\\ 6 & 1 & 1\\ -3 & 1 & 0\end{bmatrix}$:

$$C=\begin{bmatrix}3 & 0 & 1\\ 6 & 1 & 1\\ -3 & 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ -1 & 1 & 1\end{bmatrix}\times\begin{bmatrix}3 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 2\end{bmatrix}=LU,$$

where $L=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ -1 & 1 & 1\end{bmatrix}$, $U=\begin{bmatrix}3 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 2\end{bmatrix}$.

Answer: $\begin{bmatrix}3 & 0 & 1\\ 6 & 1 & 1\\ -3 & 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ -1 & 1 & 1\end{bmatrix}\times\begin{bmatrix}3 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 2\end{bmatrix}$.