Answer on Question #62621 – Math – Linear Algebra

A. Solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination.

Question

1. $\begin{cases} -x + y + 2z = 1 \\ 2x + 3y + z = -2 \\ 5x + 4y + 2z = 4 \end{cases}$

Solution

This system can be represented by the coefficient matrix: $\begin{pmatrix} -1 & 1 & 2 & | & 1 \\ 2 & 3 & 1 & | & -2 \\ 5 & 4 & 2 & | & 4 \end{pmatrix}$

Henceforth $R2 \leftarrow R2 - 2R1$ means "the second row equals the second row subtracting two times the first row".

$$\begin{pmatrix} -1 & 1 & 2 & | & 1 \\ 2 & 3 & 1 & | & -2 \\ 5 & 4 & 2 & | & 4 \end{pmatrix} \xrightarrow{R1 \leftarrow R1*(-1)} \begin{pmatrix} 1 & -1 & -2 & | & -1 \\ 2 & 3 & 1 & | & -2 \\ 5 & 4 & 2 & | & 4 \end{pmatrix} \xrightarrow[R3 \leftarrow R3-5R1]{R2 \leftarrow R2-2R1}$$

$$\to \begin{pmatrix} 1 & -1 & -2 & | & -1 \\ 0 & 5 & 5 & | & 0 \\ 0 & 9 & 12 & | & 9 \end{pmatrix} \xrightarrow[R3 \leftarrow \frac{1}{3} \cdot R3]{R2 \leftarrow \frac{1}{5} \cdot R2} \begin{pmatrix} 1 & -1 & -2 & | & -1 \\ 0 & 1 & 1 & | & 0 \\ 0 & 3 & 4 & | & 3 \end{pmatrix} \xrightarrow{R3 \leftarrow R3-3R2}$$

$$\to \begin{pmatrix} 1 & -1 & -2 & | & -1 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 1 & | & 3 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} x - y - 2z = -1 \\ y + z = 0 \\ z = 3 \end{cases} \Rightarrow \begin{cases} x = -1 + y + 2z \\ y = -z \\ z = 3 \end{cases} \Rightarrow \begin{cases} x = -1 + z \\ y = -z \\ z = 3 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = 2 \\ y = -3 \\ z = 3 \end{cases}$$

Answer: x=2, y=-3, z=3.

Question

2. $\begin{cases} 2x + 3y + z = 10 \\ 2x - 3y - 3z = 22 \\ 4x - 2y + 3z = -2 \end{cases}$

Solution

This system can be represented by the matrix: $\begin{pmatrix} 2 & 3 & 1 & | & 10 \\ 2 & -3 & -3 & | & 22 \\ 4 & -2 & 3 & | & -2 \end{pmatrix}$

$$\begin{pmatrix}2 & 3 & 1 & | & 10\\ 2 & -3 & -3 & | & 22\\ 4 & -2 & 3 & | & -2\end{pmatrix} \xrightarrow[R3\leftarrow R3-2R1]{R2\leftarrow R2-R1} \begin{pmatrix}2 & 3 & 1 & | & 10\\ 0 & -6 & -4 & | & 12\\ 0 & -8 & 1 & | & -22\end{pmatrix} \xrightarrow[R3\leftarrow -R3]{R2\leftarrow -\frac{1}{2}\cdot R2}$$

$$\rightarrow \begin{pmatrix}2 & 3 & 1 & | & 10\\ 0 & 3 & 2 & | & -6\\ 0 & 8 & -1 & | & 22\end{pmatrix} \xrightarrow{R3\leftarrow R3-\frac{8}{3}R2} \begin{pmatrix}2 & 3 & 1 & | & 10\\ 0 & 3 & 2 & | & -6\\ 0 & 0 & -\frac{19}{3} & | & 38\end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases}2x+3y+z=10\\ 3y+2z=-6\\ -\frac{19}{3}z=38\end{cases} \Rightarrow \begin{cases}x=\frac{10-3y-z}{2}\\ y=\frac{-6-2z}{3}\\ z=-6\end{cases} \Rightarrow \begin{cases}x=\frac{16+z}{2}\\ y=\frac{-6-2z}{3}\\ z=-6\end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases}x=5\\ y=2\\ z=-6\end{cases}$$

Answer: x=5, y=2, z=-6.

B. Find the inverse of the matrix.

Question

1. $\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$

Solution

$$\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}^{-1} = \frac{1}{\det(\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix})}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix} = \frac{1}{2\cdot 3-0\cdot 0}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix} = \frac{1}{6}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix} = \begin{bmatrix}\frac{3}{6} & \frac{0}{6}\\ \frac{0}{6} & \frac{2}{6}\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{bmatrix}.$$

Answer: $\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{bmatrix}$.

Question

2. $\begin{bmatrix}-1 & 1\\ 3 & -3\end{bmatrix}$

Solution

$$\det\left(\begin{bmatrix}-1 & 1\\ 3 & -3\end{bmatrix}\right) = (-1)\cdot(-3) - 3\cdot 1 = 3-3 = 0$$

It means that the inverse of the matrix doesn’t exist. Such a matrix is called “singular.

Answer: it doesn’t exist.

Question

3. $\begin{bmatrix}1 & 2\\ 3 & 7\end{bmatrix}$

Solution

$$\begin{bmatrix}1 & 2\\ 3 & 7\end{bmatrix}^{-1} = \frac{1}{\det(\begin{bmatrix}1 & 2\\ 3 & 7\end{bmatrix})}\begin{bmatrix}7 & -2\\ -3 & 1\end{bmatrix} = \frac{1}{1\cdot 7-2\cdot 3}\begin{bmatrix}7 & -2\\ -3 & 1\end{bmatrix} = \begin{bmatrix}7 & -2\\ -3 & 1\end{bmatrix}$$

Answer: $\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$.

C. Finding the inverse of the Square of a Matrix.
Direction: Compute A^-2

Question

1. $A = \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix}$

Solution

$$A^2 = \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix}^2 = \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \cdot 0 + (-2)(-1) & 0 \cdot (-2) + (-2) \cdot 3 \\ (-1) \cdot 0 + 3 \cdot (-1) & (-1) \cdot (-2) + 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ -3 & 11 \end{bmatrix}$$

$$A^{-2} = (A^2)^{-1} = \begin{bmatrix} 2 & -6 \\ -3 & 11 \end{bmatrix}^{-1} = \frac{1}{\det\left(\begin{bmatrix} 2 & -6 \\ -3 & 11 \end{bmatrix}\right)} \begin{bmatrix} 11 & 6 \\ 3 & 2 \end{bmatrix}$$

$$= \frac{1}{2 \cdot 11 - (-3) \cdot (-6)} \begin{bmatrix} 11 & 6 \\ 3 & 2 \end{bmatrix} =$$

$$= \frac{1}{4} \begin{bmatrix} 11 & 6 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{11}{4} & \frac{3}{2} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}$$

Answer: $A^{-2} = \begin{bmatrix} \frac{11}{4} & \frac{3}{2} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}$.

Question

2. $A = \begin{bmatrix} 2 & 7 \\ -5 & 6 \end{bmatrix}$

Solution

$$A^2 = \begin{bmatrix} 2 & 7 \\ -5 & 6 \end{bmatrix}^2 = \begin{bmatrix} 2 & 7 \\ -5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 2 & 7 \\ -5 & 6 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 \cdot 2 + 7 \cdot (-5) & 2 \cdot 7 + 7 \cdot 6 \\ (-5) \cdot 2 + 6 \cdot (-5) & (-5) \cdot 7 + 6 \cdot 6 \end{bmatrix} = \begin{bmatrix} -31 & 56 \\ -40 & 1 \end{bmatrix}$$

$$A^{-2} = (A^2)^{-1} = \begin{bmatrix} -31 & 56 \\ -40 & 1 \end{bmatrix}^{-1} = \frac{1}{\det\left(\begin{bmatrix} -31 & 56 \\ -40 & 1 \end{bmatrix}\right)} \begin{bmatrix} 1 & -56 \\ 40 & -31 \end{bmatrix} =$$

$$= \frac{1}{(-31) \cdot 1 - 56 \cdot (-40)} \begin{bmatrix} 1 & -56 \\ 40 & -31 \end{bmatrix} = \frac{1}{2209} \begin{bmatrix} 1 & -56 \\ 40 & -31 \end{bmatrix} = \begin{bmatrix} \frac{1}{2209} & \frac{-56}{2209} \\ \frac{40}{2209} & \frac{-31}{2209} \end{bmatrix}$$

Answer: $A^{-2} = \begin{bmatrix} \frac{1}{2209} & \frac{-56}{2209} \\ \frac{40}{2209} & \frac{-31}{2209} \end{bmatrix}$.

Question

3. $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Solution

$$A^2 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}^2 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} (-2)\cdot(-2) + 0\cdot 1 + 0\cdot 3 & (-2)\cdot 0 + 0\cdot 1 + 0\cdot 0 & (-2)\cdot 0 + 0\cdot 0 + 0\cdot 3 \\ 0\cdot(-2) + 1\cdot 0 + 0\cdot 0 & 0\cdot 0 + 1\cdot 1 + 0\cdot 0 & 0\cdot 0 + 1\cdot 0 + 0\cdot 3 \\ 0\cdot(-2) + 0\cdot 0 + 3\cdot 0 & 0\cdot 0 + 0\cdot 1 + 3\cdot 0 & 0\cdot 0 + 0\cdot 0 + 3\cdot 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 9 & | & 0 & 0 & 1 \end{bmatrix} \sim \left| R1 \leftarrow \frac{R1}{4}, R3 \leftarrow \frac{R3}{9} \right| \sim \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & \frac{1}{9} \end{bmatrix}$$

$$A^{-2} = (A^2)^{-1} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix}$$

Answer: $A^{-2} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix}$.