Answer on Question #62564 – Math – Calculus

Question

Find the Taylors expansion of f(x) = Sin x at x = 0.

Solution

If $f(x) = \sin x$, then

$$f^{(n)}(x) = \begin{cases} (-1)^k \sin x, & \text{if } n = 2k \text{ and } k = 0, 1, 2, \dots \\ (-1)^k \cos x, & \text{if } n = 2k+1 \text{ and } k = 0, 1, 2, \dots \end{cases}$$

so

$$f^{(n)}(0) = \begin{cases} 0, & \text{if } n = 2k \text{ and } k = 0, 1, 2, \dots \\ (-1)^k, & \text{if } n = 2k+1 \text{ and } k = 0, 1, 2, \dots \end{cases}$$

Therefore, the Taylors expansion of $f(x) = \sin x$ at $x = 0$ is

$$\sin x \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Since $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{k+1} x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{(-1)^k x^{2k+1}}\right| = \lim_{n\to\infty} \left|\frac{x^2}{(2k+2)(2k+3)}\right| = 0 < 1$, then the radius of convergence is $R = \infty$. Hence

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{for all } x$$

Answer:

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ for all x