Answer on Question #62201 – Math – Trigonometry

Question

Prove that

tan(10)·tan(50) + tan(50)·tan(70) + tan(70)·tan(170) = 3.

Solution

Let $\tan 10 = x$. Then

$\tan(70) = \tan(60 + 10) = \frac{\tan 60 + \tan 10}{1 - \tan 60 \tan 10} = \frac{\sqrt{3} + x}{1 - \sqrt{3}x}$;

$\tan(50) = \tan(60 - 10) = \frac{\tan 60 - \tan 10}{1 + \tan 60 \tan 10} = \frac{\sqrt{3} - x}{1 + \sqrt{3}x}$;

$\tan(170) = \tan(180 - 10) = -\tan(10) = -x$;

$$\tan(10) \cdot \tan(50) + \tan(50) \cdot \tan(70) + \tan(70) \cdot \tan(170) =$$

$$= x\frac{\sqrt{3} - x}{1 + \sqrt{3}x} + \frac{\sqrt{3} - x}{1 + \sqrt{3}x} \cdot \frac{\sqrt{3} + x}{1 - \sqrt{3}x} - x\frac{\sqrt{3} + x}{1 - \sqrt{3}x} =$$

$$= x\frac{(\sqrt{3} - x)(1 - \sqrt{3}x) - (\sqrt{3} + x)(1 + \sqrt{3}x)}{1 - 3x^2} + \frac{3 - x^2}{1 - 3x^2} =$$

$$= x\frac{\sqrt{3} - x - 3x + \sqrt{3}x^2 - \sqrt{3} - x - 3x - \sqrt{3}x^2}{1 - 3x^2} + \frac{3 - x^2}{1 - 3x^2} = \frac{-8x^2 + 3 - x^2}{1 - 3x^2} =$$

$= \frac{3 - 9x^2}{1 - 3x^2} = 3$, when $1 - 3x^2 \neq 0 => x \neq \pm\frac{1}{\sqrt{3}}$.

Note that

$\tan(10) \approx 0.6484$, $\tan(10^{\circ}) \approx 0.1763$, $\frac{1}{\sqrt{3}} \approx 0.5774$.

Since

$\tan 10 \neq \pm\frac{1}{\sqrt{3}}$.

Therefore, the statement is proved.