Answer on Question #62146 – Math – Trigonometry

Question

If

$$\begin{cases} x = a\sin Q + b\cos Q, \\ y = a\cos Q + b\sin Q. \end{cases}$$

Prove that $(x^2 + y^2)(a^2 + b^2) - 4abxy = (a^2 - b^2)^2$

Solution

$$x^2 + y^2 = (a\sin Q + b\cos Q)^2 + (a\cos Q + b\sin Q)^2 =$$
$$= a^2(\sin Q)^2 + 2ab\sin Q\cos Q + b^2(\cos Q)^2 + 2ab\sin Q\cos Q +$$
$$+ b^2(\sin Q)^2 + 2ab\sin Q\cos Q + a^2(\cos Q)^2 = a^2 + b^2 + 4ab\sin Q\cos Q;$$

$$xy = a^2\sin Q\cos Q + ab(\sin Q)^2 + ab(\cos Q)^2 + b^2\sin Q\cos Q =$$
$$= ab + (a^2 + b^2)\sin Q\cos Q\,;$$

$$(x^2 + y^2)(a^2 + b^2) - 4abxy =$$
$$= (a^2 + b^2 + 4ab\sin Q\cos Q)(a^2 + b^2) - 4ab(ab + (a^2 + b^2)\sin Q\cos Q) =$$
$$= (a^2 + b^2)^2 - 4a^2b^2 + (4ab\sin Q\cos Q - 4ab\sin Q\cos Q)(a^2 + b^2) =$$
$$= a^4 + 2a^2b^2 + b^4 - 4a^2b^2 = a^4 - 2a^2b^2 + b^4 = (a^2 - b^2)^2,$$

which proves that
$(x^2 + y^2)(a^2 + b^2) - 4abxy = (a^2 - b^2)^2.$