Answer on Question #62080 – Math – Linear Algebra

Question

Let (x_1, x_2, x_3) and (y_1, y_2, y_3) represent the coordinates with respect to the bases

$B_1 = \{(1,0,0),(0,1,0),(0,0,1)\}$, $B_2 = \{(1,0,0),(0,1,2),(0,2,1)\}$.

If

$Q(X) = 2x_1^2 + 2x_1x_2 - 2x_2x_3 + x_2^2 + x_3^2$,

find the representation of Q in terms of (y_1, y_2, y_3).

Solution

First of all, let us find the transition matrix $C = C_{B_1 \to B_2}$.

The following equalities are true:

$(1,0,0) = 1 \cdot (1,0,0) + 0 \cdot (0,1,0) + 0 \cdot (0,0,1)$;

$(0,1,2) = 0 \cdot (1,0,0) + 1 \cdot (0,1,0) + 2 \cdot (0,0,1)$;

$(0,2,1) = 0 \cdot (1,0,0) + 2 \cdot (0,1,0) + 1 \cdot (0,0,1)$.

So

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

Given $Q(X) = 2x_1^2 + 2x_1x_2 - 2x_2x_3 + x_2^2 + x_3^2$, the matrix of $Q(X)$ in the base B_1:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Then the matrix of the form Q in the base B_2 is equal to

$$B = C^T A C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}.$$

So the form Q in the base B_2 will be

$Q(Y) = 2y_1^2 + y_2^2 + y_3^2 + 2y_1y_2 + 4y_1y_3 - 2y_2y_3$.

Answer: $Q(Y) = 2y_1^2 + y_2^2 + y_3^2 + 2y_1y_2 + 4y_1y_3 - 2y_2y_3$.