Answer on Question #62052 – Math – Calculus

Question

Trace the curve $y^2(x^2-9) = x^4$ by stating all the properties used in tracing.

Solution

Symmetry

The curve is symmetric with respect to x-axis and y-axis, because it includes x and y in even degrees. So we can trace its graph only in the first quadrant, where $x \geq 0, y \geq 0$.

The graphs in the other quadrants can be obtained using symmetry.

The curve is symmetric with respect to origin, because when we replace both x with -x and y with -y the equation will be the same.

Let $x \geq 0$ and $y \geq 0$. It follows from $y^2 = \frac{x^4}{x^2-9}$ that $y = \frac{x^2}{\sqrt{x^2-9}}$, when $y \geq 0$.

Intercepts

y=0 if and only if x=0.

Intercepts: (0, 0)

Domain

If $y^2 = \frac{x^4}{x^2-9}$, then $x \neq 3$ and $x \neq -3$.

Besides, $y^2 \geq 0$ and $x^4 \geq 0$, then $x^2 - 9 > 0$, hence $x > 3$ or $x < -3$. It was found that $x = 0$ if and only if $y = 0$.

Asymptotes

Let $x > 3$:

$$y = \frac{x^2}{\sqrt{x^2-9}}$$

$\lim_{x\to 3^+} \frac{x^2}{\sqrt{x^2-9}} = \infty$, hence x=3 is a vertical asymptote. Similarly x=-3 will be a vertical asymptote.

$\lim_{x\to\infty} \frac{x^2}{\sqrt{x^2-9}} = \infty$, $k = \lim_{x\to\infty} \frac{x^2}{x\sqrt{x^2-9}} = 1$,

$$b = \lim_{x\to\infty}\left(\frac{x^2}{\sqrt{x^2-9}} - kx\right) = \lim_{x\to\infty}\left(\frac{x^2}{\sqrt{x^2-9}} - x\right) = \lim_{x\to\infty}\left(\frac{x^2-9+9}{\sqrt{x^2-9}} - x\right) = \lim_{x\to\infty}\left(\sqrt{x^2-9} + \frac{9}{\sqrt{x^2-9}} - x\right) =$$

$$= \lim_{x\to\infty}\left(\sqrt{x^2-9} - x + \frac{9}{\sqrt{x^2-9}}\right) = \lim_{x\to\infty}\left(\frac{x^2-9-x^2}{\sqrt{x^2-9}+x} + \frac{9}{\sqrt{x^2-9}}\right) = \lim_{x\to\infty}\left(\frac{-9}{\sqrt{x^2-9}+x} + \frac{9}{\sqrt{x^2-9}}\right) = 0.$$

Hence $y = kx + b$, i. e., y=x is an oblique asymptote.

Critical points:

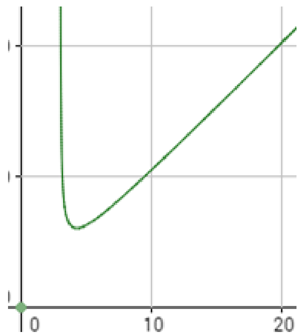
$$\frac{d}{dx}\left(\frac{x^2}{\sqrt{x^2-9}}\right)=\frac{d}{dx}\left(\frac{x^2-9+9}{\sqrt{x^2-9}}\right)=\frac{d}{dx}\left(\sqrt{x^2-9}+\frac{9}{\sqrt{x^2-9}}\right)=\frac{2x}{2\sqrt{x^2-9}}-\frac{9}{2(x^2-9)^{\frac{3}{2}}}\times 2x=\frac{x(x^2-18)}{(x^2-9)^{\frac{3}{2}}}$$

Critical point at x=$3\sqrt{2}$ (local minimum)

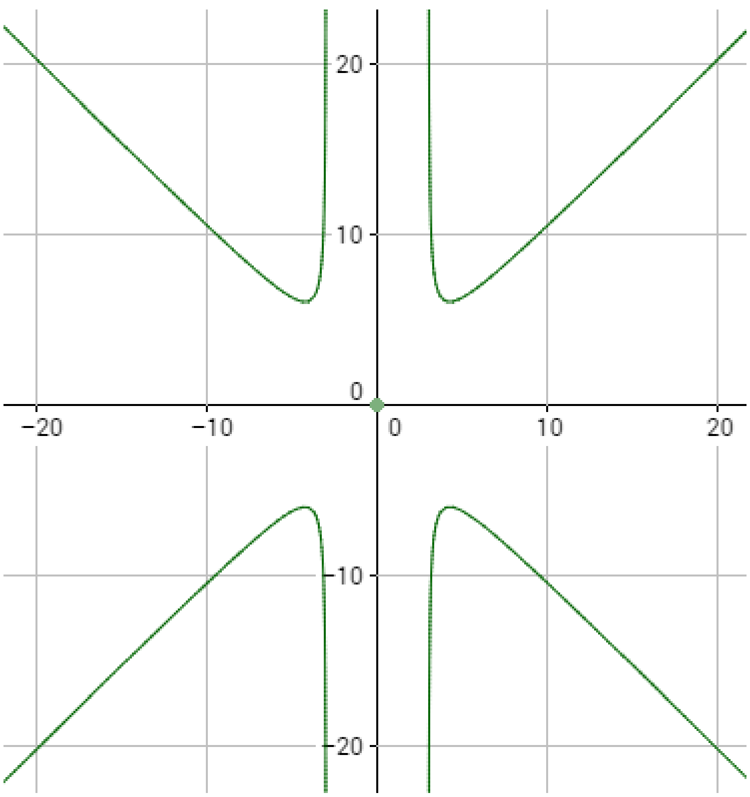
y$(3\sqrt{2})=6$

$\frac{d^2}{dx^2}\left(\frac{x^2}{\sqrt{x^2-9}}\right)=\frac{9(x^2+18)}{(x^2-9)^{\frac{5}{2}}}$>0 if x>3, hence function is concave up

Therefore, the graph in the first quadrant will look as follows:



Hence, the graph for the curve is



The trace for the whole curve.

Using the symmetries

Domain: $x \in (-\infty, -3) \cup \{0\} \cup (3, \infty)$.

Range: $y \in (-\infty, -6) \cup \{0\} \cup (6, \infty)$.

Intercept: (0, 0).

Asymptotes: y = x, y = –x, x = 3, x = –3.

Critical points: $(3\sqrt{2}, 6), (3\sqrt{2}, -6), (-3\sqrt{2}, 6), (-3\sqrt{2}, -6)$.