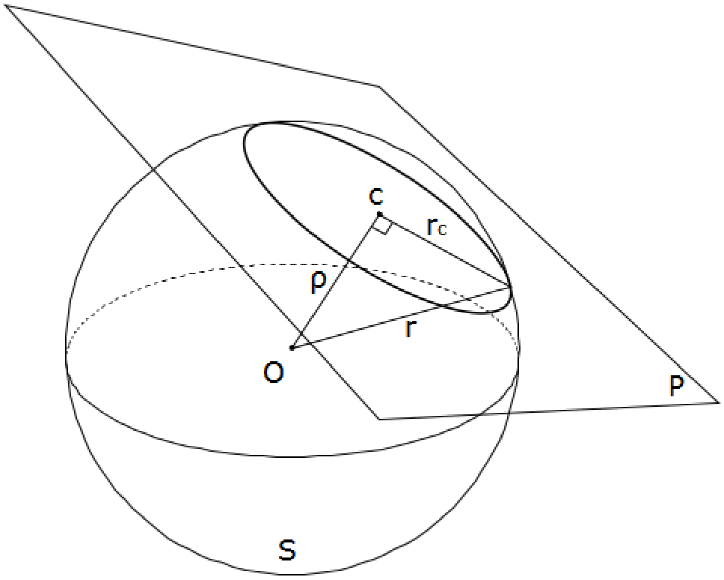
Answer on Question #62051 – Math – Analytic Geometry

Question

Find the radius and the centre of the circular section of the sphere | r| = 4 cut off by the plane r*(2i-j+4k) = 3.

Solution



Let S be the sphere in R^3 with center $O(x_0, y_0, z_0)$ and radius r, and let P be the plane with equation $Ax + By + Cz = D$, so that $\vec{n} = (A, B, C)$ is a normal vector of P.

If P_0 is an arbitrary point on P, the *signed* distance from the center of the sphere O to the plane P is

$$\rho = \frac{(O - P_0)\vec{n}}{||\vec{n}||} = \frac{Ax_0 + B\,y_0 + Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}}.$$

The intersection $S \cap P$ is a circle if and only if $-r < \rho < r$, and in that case, the circle has radius $r_c = \sqrt{r^2 - \rho^2}$ and center

$$c = O + \rho \cdot \frac{\vec{n}}{||\vec{n}||} = (x_0, y_0, z_0) + \rho \cdot \frac{(A, B, C)}{\sqrt{A^2 + B^2 + C^2}}.$$

In our case:

$O(0,0,0); |r| = 4;\ \ 2x - y + 4z = 3$

$$S = \{(x, y, z): x^2 + y^2 + z^2 = 16\}, \qquad P = \{(x, y, z): 2x - y + 4z = 3\}.$$

$$\rho = \frac{Ax_0 + B\,y_0 + Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}} = \frac{2 \cdot 0 - 1\ \cdot 0 + 4 \cdot 0 - 3}{\sqrt{2^2 + (-1)^2 + 4^2}} = -\frac{3}{\sqrt{21}}.$$

$$r_c = \sqrt{r^2 - \rho^2} = \sqrt{4^2 - (-\frac{3}{\sqrt{21}})^2} = \sqrt{\frac{109}{7}} \approx 3.946.$$

$$c = O + \rho \cdot \frac{\vec{n}}{||\vec{n}||} = (0,0,0) + \left(-\frac{3}{\sqrt{21}}\right) \cdot \frac{(2,-1,4)}{\sqrt{2^2 + (-1)^2 + 4^2}} = \left(-\frac{2}{7}, \frac{1}{7}, -\frac{4}{7}\right).$$

Answer: $\sqrt{\frac{109}{7}}$, $\left(-\frac{2}{7}, \frac{1}{7}, -\frac{4}{7}\right)$.