Answer on Question #62049 – Math – Linear Algebra

Question

Find the nature of the conic
$2x^2 + xy - y^2 - 6 = 0.$

Solution

The general equation of the second degree in two variables is

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

$$A = 2; 2B = 1; C = -1; 2D = 0; 2E = 0; F = -6.$$

a) Simplification by rotation of coordinate system.
With a suitable rotation of the coordinate system the term in xy can be eliminated from any second degree equation. Under a rotation of the coordinate system about its origin by an angle of θ degrees

$$\tan 2\theta = \frac{2B}{A - C} = \frac{1}{2 - (-1)} = \frac{1}{3}.$$

$x = x' \cos\theta - y' \sin\theta;$

$y = x' \sin\theta + y' \cos\theta;$

$$\cos 2\theta = \frac{1}{\sqrt{1 + (\tan 2\theta)^2}} = \frac{3}{\sqrt{10}};$$

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \sqrt{\frac{\sqrt{10} - 3}{2\sqrt{10}}};$$

$$\cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \sqrt{\frac{\sqrt{10} + 3}{2\sqrt{10}}}.$$

$$x = x'\sqrt{\frac{\sqrt{10} + 3}{2\sqrt{10}}} - y'\sqrt{\frac{\sqrt{10} - 3}{2\sqrt{10}}};$$

$$y = x'\sqrt{\frac{\sqrt{10} - 3}{2\sqrt{10}}} + y'\sqrt{\frac{\sqrt{10} + 3}{2\sqrt{10}}};$$

the general equation of the second degree becomes

$$A'x'^2 + 2B'x'y' + C'y'^2 + 2D'x' + 2E'y' + F' = 0$$

where

$A' = A(\cos\theta)^2 + 2B\sin\theta\cos\theta + C(\sin\theta)^2$

$B' = (C - A)\sin\theta\cos\theta + B((\cos\theta)^2 - (\sin\theta)^2) = \frac{1}{2}(C - A)\sin 2\theta + B\cos 2\theta$

$C' = A(\sin\theta)^2 - 2B\sin\theta\cos\theta + C(\cos\theta)^2$

$D' = D\cos\theta + E\sin\theta$

$E' = E\cos\theta - D\sin\theta$

$F' = F$

$$A' = 2\left(\frac{\sqrt{10}+3}{2\sqrt{10}}\right) + \sqrt{(\frac{\sqrt{10}-3}{2\sqrt{10}})(\frac{\sqrt{10}+3}{2\sqrt{10}})} - \left(\frac{\sqrt{10}-3}{2\sqrt{10}}\right) = \frac{1+\sqrt{10}}{2}$$

$$B' = (-1-2)\sqrt{\left(\frac{\sqrt{10}-3}{2\sqrt{10}}\right)\left(\frac{\sqrt{10}+3}{2\sqrt{10}}\right)} + \frac{1}{2}\left(\frac{\sqrt{10}+3}{2\sqrt{10}} - \frac{\sqrt{10}-3}{2\sqrt{10}}\right) = 0$$

$$C' = 2\left(\frac{\sqrt{10}-3}{2\sqrt{10}}\right) + \sqrt{(\frac{\sqrt{10}-3}{2\sqrt{10}})(\frac{\sqrt{10}+3}{2\sqrt{10}})} - \left(\frac{\sqrt{10}+3}{2\sqrt{10}}\right) = \frac{1-\sqrt{10}}{2}$$

$D' = 0$

$E' = 0$

$F' = -6$

$$\frac{1+\sqrt{10}}{2}x'^2 + \frac{1-\sqrt{10}}{2}y'^2 - 6 = 0$$

$$\frac{x'^2}{[\sqrt{12\cdot(\sqrt{10}-1)}]^2} - \frac{y'^2}{[\sqrt{12\cdot(\sqrt{10}+1)}]^2} = 1$$

This is the equation of hyperbola:

$$a = \sqrt{12\cdot(\sqrt{10}-1)};\ b = \sqrt{12\cdot(\sqrt{10}+1)}.\, c^2 = a^2 + b^2$$

$$= 12\cdot\left(\sqrt{10}-1+\sqrt{10}+1\right) = 24\cdot\sqrt{10}.$$

Answer: hyperbola.