Answer on Question #62033 – Math – Geometry

Question

Calculate the volume of the tetrahedron whose vertices are the points A = (3, 2, 1), B = (1, 2, 4), C = (4, 0, 3) and D = (1, 1, 7).

Solution

The absolute value $\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|$ of the scalar triple product of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is the volume of the parallelepiped spanned by $\vec{a}$, $\overleftrightarrow{b}$ and $\vec{c}$. A tetrahedron is a pyramid, so the formula for the volume is

$$V = \frac{1}{3}Ah,$$

where A is base area (area triangle), h is a height of pyramid. The basis of the parallelepiped is a parallelogram, whose area is twice the area of the triangle formed by the same vectors. The height of the pyramid and parallelepiped built on the same vectors are the same.

Hence

$$\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right| = 6V_{tetrahedron}$$

and

$$V_{tetrahedron} = \frac{1}{6}\left|\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\cdot\overrightarrow{AD}\right|.$$

Vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are given as follows:

$\overrightarrow{AB} = (x_B - x_A, y_B - y_A, z_B - z_A) = (1-3, 2-2, 4-1) = (-2,0,3);$

$\overrightarrow{AC} = (x_C - x_A, y_C - y_A, z_C - z_A) = (4-3, 0-2, 3-1) = (1,-2,2);$

$\overrightarrow{AD} = (x_D - x_A, y_D - y_A, z_D - z_A) = (1-3, 1-2, 7-1) = (-2,-1,6).$

The scalar triple product $\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\cdot\overrightarrow{AD}$ is

$$\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\cdot\overrightarrow{AD} = \begin{vmatrix} -2 & 0 & 3 \\ 1 & -2 & 2 \\ -2 & -1 & 6 \end{vmatrix} = (-2)\cdot(-2)\cdot 6 + 0\cdot 2\cdot(-2) +$$

$+1\cdot(-1)\cdot 3 - (-2)\cdot(-2)\cdot 3 - (-1)\cdot 2\cdot(-2) - 1\cdot 0\cdot 6 = 24 + 0 - 3 -$
$-12 - 4 - 0 = 24 - 19 = 5,$

hence

$$\left|\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\cdot\overrightarrow{AD}\right| = 5.$$

Therefore, the volume of the tetrahedron equals

$$V_{tetrahedron} = \frac{1}{6}\cdot 5 = \frac{5}{6}.$$

Answer: $\frac{5}{6}$.