Answer on Question #61931 – Math – Linear Algebra

Question

Show that an orthogonal map of the plane is either a reflection, or a rotation.

Solution

If $T: R^2 \to R^2$ is an orthogonal linear operator, then the standard matrix for T can be expressed in the form

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$(R_\theta \; rotates \; vectors \; by \; \theta \; radians, counterclockwise)$$

or

$$H_{\frac{\theta}{2}} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

The determinant can be used to distinguish between the two cases:

$$\det R_\theta = \cos^2\theta + \sin^2\theta = 1$$

$$\det H_{\frac{\theta}{2}} = -(\cos^2\theta + \sin^2\theta) = -1.$$

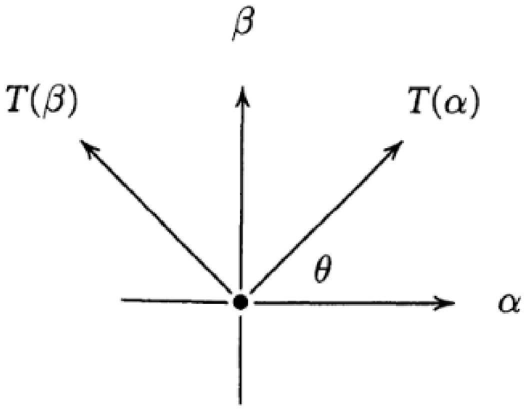
Thus, a 2x2 orthogonal matrix represents a rotation about the origin if det = 1 and a reflection about a line through the origin if det = -1. This sentence may be a definition of an orthogonal map of plane. If the definition of 2×2 matrix is $A^T A = I$, where I is an identity matrix, then the determinant of an orthogonal matrix is equal to 1 or -1, because using properties of determinants and equality $A^T A = I$, obtain $\det(A) = det(A^T)$ and

$1 = det(I) = det(A^T A) = det(A^T)det(A) = (\det(A))^2$, hence $\det(A) = 1$ or $\det(A) = -1$.

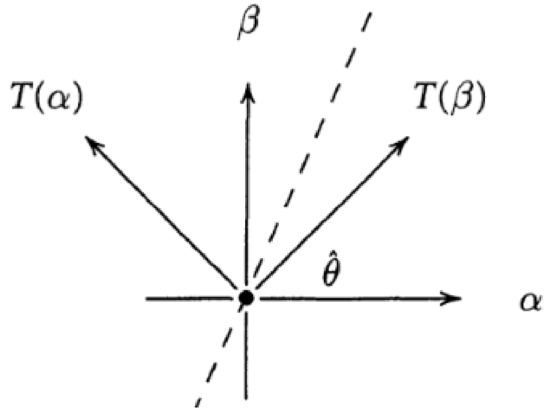
Lemma. Every orthogonal transformation of the Euclidean plane is a reflection or a rotation.

We consider the effect of an orthogonal transformation *T* on the two vectors $\alpha = (1{,}0)$ and

$\beta = (0{,}1)$. The orthogonality of *T* forces $T(\alpha)$ and $T(\beta)$ also to be of unit length and orthogonal to each other. Let θ and $\hat{\theta}$ be the counter-clockwise angles formed by $T(\alpha)$ and $T(\beta)$ with the *x*-axis. Then $\hat{\theta} = \theta \pm \frac{\pi}{2}$. In the case $\hat{\theta} = \theta + \frac{\pi}{2}$, *T* is a rotation through angle θ. In the case $\hat{\theta} = \theta - \frac{\pi}{2}$, *T* is a reflection through the dashed line forming an angle of $\frac{\theta}{2}$ with the *x*-axis. In the pictures we have



β
$T(\beta)$
$T(\alpha)$
θ
α



β
$T(\alpha)$
$T(\beta)$
$\hat{\theta}$
α