Answer on Question #61910 – Math – Linear Algebra

Question

Apply the Gram-Schmidt diagonalisation process to find an orthonormal basis for the subspace of C^4 generated by the vectors

{(1,−i,0,1),(−1,0, i,0),(−i,0,1,−1)}.

Solution

For vectors in C^4 : $< x, y >= x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3 + x_4\bar{y}_4,$

Basis:

$u_1 = v_1 = (1, -i, 0,1);$

$< u_1, u_1 >= 1^2 + 1^2 + 0^2 + 1^2 = 3;$

$\|u_1\| = \sqrt{\langle u_1, u_1 \rangle} = \sqrt{3};$

$v_2 = (-1,0, i, 0);$

$< v_2, u_1 >= -1 \cdot 1 + 0 \cdot i + i \cdot 0 + 0 \cdot 1 = -1;$

$u_2 = v_2 - proj_{u_1} v_2 = v_2 - \frac{<v_2,u_1>}{<u_1,u_1>} u_1 = (-1,0, i, 0) - (-\frac{1}{3})(1, -i, 0,1)=$

$= \left(-1 + \frac{1}{3},\ 0 - \frac{i}{3}, i, \frac{1}{3}\right) = \left(-\frac{2}{3}, -\frac{i}{3}, i, \frac{1}{3}\right);$

$< u_2, u_2 >= \left(-\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + 1^2 + \left(\frac{1}{3}\right)^2 = \frac{15}{9} = \frac{5}{3};$

$\|u_2\| = \sqrt{\langle u_2, u_2 \rangle} = \sqrt{\frac{5}{3}};$

$< v_3, u_1 >= -i \cdot 1 + 0 \cdot i + 1 \cdot 0 + (-1) \cdot 1 = -i - 1 = -(i + 1);$

$< v_3, u_2 >= -i \cdot \left(-\frac{2}{3}\right) + 0 \cdot \frac{i}{3} + 1 \cdot (-i) + (-1) \cdot \frac{1}{3} = -\frac{i}{3} - \frac{1}{3} = -\frac{i+1}{3};$

$u_3 = v_3 - proj_{u_1} v_3 - proj_{u_2} v_3 = v_3 - \frac{<v_3,u_1>}{<u_1,u_1>} u_1 - \frac{<v_3,u_2>}{<u_2,u_2>} u_2 = (-i, 0,1, -1) +$

$+ \frac{i+1}{3}(1, -i, 0,1) + \frac{i+1}{3\times\frac{5}{3}}\left(-\frac{2}{3}, -\frac{i}{3}, i, \frac{1}{3}\right) = (-i, 0,1, -1) + \frac{i+1}{3}(1, -i, 0,1) +$

$+ \frac{i+1}{15}(-2, -i, 3i, 1) = \left(-i + \frac{i+1}{3} - \frac{2i+2}{15},\ 0 - i \times \frac{i+1}{3} - i \times \frac{i+1}{15}, 1 + 0 + 3i \times\right.$

$\left.\frac{i+1}{15}, -1 + + \frac{i+1}{3} + \frac{i+1}{15}\right) = \left(\frac{-15i+5i+5-2i-2}{15},\ \frac{-5i^2-5i-i^2-i}{15}, 1 + \frac{i^2+i}{5}, \frac{-15+5i+5+i+1}{15}\right) =$

$= \left(\frac{-12i+3}{15}, \frac{6-6i}{15}, \frac{4+i}{5}, \frac{6i-9}{15}\right) = \left(\frac{1-4i}{5}, \frac{2-2i}{5}, \frac{4+i}{5}, \frac{2i-3}{5}\right);$

$< u_3, u_3 >=$

$$= \frac{1}{5^2}\big((1-4i)(1+4i) + (2-2i)(2+2i) + (4-i)(4+i) + (2i-3)(2i+3)\big) =$$

$= \frac{1}{25}(1 + 16 + 4 + 4 + 16 + 1 - 4 - 9) = \frac{29}{25};$

$\|u_3\| = \sqrt{\langle u_3, u_3\rangle} = \sqrt{\frac{29}{25}} = \frac{\sqrt{29}}{5}.$

Checking:

$\langle u_2, u_1\rangle = \left(-\frac{2}{3}, -\frac{i}{3}, i, \frac{1}{3}\right) \cdot (1, -i, 0, 1) = -\frac{2}{3} + \frac{1}{3} + \frac{1}{3} = 0;$

$\langle u_3, u_1\rangle = \left(\frac{1-4i}{5}, \frac{2-2i}{5}, \frac{4+i}{5}, \frac{2i-3}{5}\right) \cdot (1, -i, 0, 1) = \frac{1}{5}\big(1 - 4i + i(2-2i) + (2i-3)\big) =$

$= 0;$

$\langle u_3, u_2\rangle = \left(\frac{1-4i}{5}, \frac{2-2i}{5}, \frac{4+i}{5}, \frac{2i-3}{5}\right) \cdot \left(-\frac{2}{3}, -\frac{i}{3}, i, \frac{1}{3}\right) = \frac{1}{15}(-2(1-4i) + i(2-2i) -$

$-3i(4+i) + (2i-3)) = 0.$

So vectors u_1, u_2, u_3 are orthogonal.

Orthonormal basis:

$e_1 = \frac{u_1}{||u_1||} = \frac{1}{\sqrt{3}}(1, -i, 0, 1) = \left(\frac{1}{\sqrt{3}}, -\frac{i}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right)$

$e_2 = \frac{u_2}{||u_2||} = \sqrt{\frac{3}{5}}\left(-\frac{2}{3}, -\frac{i}{3}, i, \frac{1}{3}\right) = \left(-\frac{2}{\sqrt{15}}, -\frac{i}{\sqrt{15}}, \frac{3i}{\sqrt{15}}, \frac{1}{\sqrt{15}}\right)$

$e_3 = \frac{u_3}{||u_3||} = \frac{5}{\sqrt{29}}\left(\frac{1-4i}{5}, \frac{2-2i}{5}, \frac{4+i}{5}, \frac{-3+2i}{5}\right) = \left(\frac{1-4i}{\sqrt{29}}, \frac{2-2i}{\sqrt{29}}, \frac{4+i}{\sqrt{29}}, \frac{-3+2i}{\sqrt{29}}\right).$

Answer: $\left\{\left(\frac{1}{\sqrt{3}}, -\frac{i}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right), \left(-\frac{2}{\sqrt{15}}, -\frac{i}{\sqrt{15}}, \frac{3i}{\sqrt{15}}, \frac{1}{\sqrt{15}}\right), \left(\frac{1-4i}{\sqrt{29}}, \frac{2-2i}{\sqrt{29}}, \frac{4+i}{\sqrt{29}}, \frac{-3+2i}{\sqrt{29}}\right)\right\}.$