Answer on Question #61909 – Math – Linear Algebra

Question

Let T : P1 → P2 be defined by

T(a+bx) = b+ax+(a−b)x^2.

Check that T is a linear transformation. Find the matrix of the transformation with respect to the ordered bases B1 = {1,x} and B2 = {x^2,x^2+x,x^2+x+1}. Find the kernel of T. Further check that the range of T is {a+bx+cx^2 ∈ P2 | a+c = b}.

Solution

Let $T(a+bx) = b + ax + (a-b)x^2$, then $T(ac+bdx) = bd + acx + (ac-bd)x^2$,

$cT(a) + d\big(T(bx)\big) = c(ax + ax^2) + d(b - bx^2) = bd + acx + (ac - bd)x^2 = T(ac + bdx)$

So, T is a linear transformation.

If $B_1 = \{1, x\} = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ and $B_2 = \{x^2,\ x^2+x,\ x^2+x+1\} = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$, then

$$T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = T(1 + 0\cdot x) = x + x^2 = 0\cdot x^2 + 1\cdot(x^2+x) + 0\cdot(x^2+x+1) = 1\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix};$$

$$T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = T(0 + 1\cdot x) = 1 - x^2 = -x^2 - (x^2+x) + (x^2+x+1) = -\begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\1\end{pmatrix}.$$

The matrix of the transformation T is $A = \begin{pmatrix}0 & -1\\1 & -1\\0 & 1\end{pmatrix}$.

The following system determines the kernel of T:

$$\begin{pmatrix}0 & -1\\1 & -1\\0 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \to -b = 0; b = 0; a - b = 0 \to \begin{cases}a = 0\\b = 0\end{cases}.$$

Thus, the kernel is $Ker = \begin{pmatrix}0\\0\end{pmatrix}$, which is the same as $0 + 0\cdot x = 0$.

The range is $T(\alpha + \beta x) = \beta + \alpha x + (\alpha - \beta)x^2 = a + bx + cx^2$,

where $a = \beta; b = \alpha; c = \alpha - \beta$.

Thus, $a + c = \beta + \alpha - \beta = \alpha = b$ and the range is $\{a + bx + cx^2 \in P_2 | a + c = b\}$.

Answer: $\begin{pmatrix}0 & -1\\1 & -1\\0 & 1\end{pmatrix}$, 0, $\{a + bx + cx^2 \in P_2 | a + c = b\}$.