Answer on Question #61907 – Math – Linear Algebra

Question

Find the orthogonal canonical reduction of the quadratic form −x2 + y2 + z2− 6xy− 6xz+ 2yz . Also, find its principal axes, rank and signature of the quadratic form.

$$-x^2 + y^2 + z^2 - 6xy - 6xz + 2yz$$

Solution

The matrix of the quadratic form $-x^2 + y^2 + z^2 - 6xy - 6xz + 2yz$ is

$$A = \begin{pmatrix} -1 & -3 & -3 \\ -3 & 1 & 1 \\ -3 & 1 & 1 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} -1-\lambda & -3 & -3 \\ -3 & 1-\lambda & 1 \\ -3 & 1 & 1-\lambda \end{vmatrix} = 0,$$

hence

$\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0,$

where

$$D_1 = trace(A) = -1 + 1 + 1 = 1;$$
$$D_2 = \begin{vmatrix} -1 & -3 \\ -3 & 1 \end{vmatrix} + \begin{vmatrix} -1 & -3 \\ -3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} =$$
$$= \big(-1 \cdot 1 - (-3) \cdot (-3)\big) + (-1 \cdot 1 - (-3) \cdot (-3)) + (1 \cdot 1 - 1 \cdot 1) =$$
$$= (-1 - 9) + (-1 - 9) + (1 - 1) = -10 - 10 = -20;$$

$D_3 = \det(A) = \begin{vmatrix} -1 & -3 & -3 \\ -3 & 1 & 1 \\ -3 & 1 & 1 \end{vmatrix} = 0,$ because the second and the third rows of the matrix A are equal.

Hence

$$\lambda^3 - \lambda^2 - 20\lambda = 0.$$

It is easy to see that $\lambda = 0$ is the root of the equation.

$$\lambda^3 - \lambda^2 - 20\lambda = \lambda(\lambda^2 - \lambda - 20) = \lambda(\lambda + 4)(\lambda - 5).$$

Therefore, the orthogonal canonical reduction of the quadratic form

$-x^2 + y^2 + z^2 - 6xy - 6xz + 2yz$ is

$$Q = (x' \quad y' \quad z')\begin{pmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = -4x'^2 + 5z'^2. \textbf{ (1)}$$

If $\lambda = -4$, then

$$\begin{pmatrix} -1+4 & -3 & -3 \\ -3 & 1+4 & 1 \\ -3 & 1 & 1+4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 3 & -3 & -3 \\ -3 & 5 & 1 \\ -3 & 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

The principal axis is $\frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

If $\lambda = 0$, then

$$\begin{pmatrix} -1-0 & -3 & -3 \\ -3 & 1-0 & 1 \\ -3 & 1 & 1-0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} -1 & -3 & -3 \\ -3 & 1 & 1 \\ -3 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

The principal axis is $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

If $\lambda = 5$, then

$$\begin{pmatrix} -1-5 & -3 & -3 \\ -3 & 1-5 & 1 \\ -3 & 1 & 1-5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} -6 & -3 & -3 \\ -3 & -4 & 1 \\ -3 & 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

The principal axis is $\frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

The rank is the total number r of square terms (both positive and negative). Using (1) we come to conclusion that the rank is $r = 2$.

Some authors assume the signature to be $n_+ - n_-$, where n_+ is the number of positive coefficients in the canonical representation of the form, n_- is the number of negative coefficients in the canonical representation of the form. Using (1) we come to conclusion that the signature is $1 - 1 = 0$.

Other authors assume the signature to be (n_+, n_-). Using (1) we come to conclusion that the signature is $(1,1)$ in this case.

Answer: $-4x'^2 + 5z'^2; \left\{\frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}; r = 2$; either 0 or (1,1) (according to different definitions).