# Answer on Question #61896 – Math – Linear Algebra

## Question

Let (x1, x2, x3) and (y1, y2, y3) represent the coordinates with respect to the bases
B1 = {(1,0,0),(0,1,0),(0,0,1)}, B2 = {(1,0,0),(0,1,2),(0,2,1)}.
If Q(X) = 2x21 +2x1x2 −2x2x3 +x22 +x23, find the representation of Q in terms of (y1, y2, y3).

## Solution

Given bases:

$$B_1: \begin{cases} x_1 = \langle 1; 0; 0 \rangle \\ x_2 = \langle 0; 1; 0 \rangle \\ x_3 = \langle 0; 0; 1 \rangle \end{cases}$$

$$B_2: \begin{cases} y_1 = \langle 1; 0; 0 \rangle \\ y_2 = \langle 0; 1; 2 \rangle \\ y_3 = \langle 0; 2; 1 \rangle \end{cases}$$

and the following polynomial:

$$Q(x) = 2x_1^2 + 2x_1x_2 - 2x_2x_3 + x_2^2 + x_3^2, \quad \textbf{(1)}$$

let's express variable "x" in terms of "y":

$$\begin{cases} y_1 = x_1 \\ y_2 = x_2 + 2x_3 \\ y_3 = 2x_2 + x_3 \end{cases} \xRightarrow{R_2 \leftarrow R_2 - 2R_3} \begin{cases} y_1 = x_1 \\ y_2 - 2y_3 = -3x_2 \\ y_3 = 2x_2 + x_3 \end{cases} \xRightarrow{R_2 \leftarrow R_2 \div (-3)} \begin{cases} x_1 = y_1 \\ x_2 = \frac{2}{3}y_3 - \frac{1}{3}y_2 \\ 2x_2 + x_3 = y_3 \end{cases}$$

$$\xRightarrow{R_3 \leftarrow R_3 - 2R_2} \begin{cases} x_1 = y_1 \\ x_2 = \frac{2}{3}y_3 - \frac{1}{3}y_2 \\ x_3 = y_3 - \frac{4}{3}y_3 + \frac{2}{3}y_2 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 \\ x_2 = \frac{2}{3}y_3 - \frac{1}{3}y_2 \\ x_3 = \frac{2}{3}y_2 - \frac{1}{3}y_3 \end{cases} \quad \textbf{(2)}$$

**Method 1**

Substituting expressions for $x_1$, $x_2$, $x_3$ from (2) into (1)

$$Q(x) = 2x_1^2 + 2x_1x_2 - 2x_2x_3 + x_2^2 + x_3^2 = 2x_1^2 + 2x_1x_2 + (x_2 - x_3)^2 =$$

$$= 2y_1^2 + 2y_1\left(\frac{2}{3}y_3 - \frac{1}{3}y_2\right) + \left(\frac{2}{3}y_3 - \frac{1}{3}y_2 - \frac{2}{3}y_2 + \frac{1}{3}y_3\right)^2 = 2y_1^2 + \frac{4}{3}y_1y_3 - \frac{2}{3}y_1y_2 + (y_3 - y_2)^2 =$$

$$= 2y_1^2 + \frac{4}{3}y_1y_3 - \frac{2}{3}y_1y_2 + y_2^2 - 2y_2y_3 + y_3^2.$$

**Method 2**

It follows from (2) that

$$x = Uy, \quad \textbf{(3)}$$

where $x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$, $U=\begin{pmatrix}1&0&0\\0&-\frac{1}{3}&\frac{2}{3}\\0&\frac{2}{3}&-\frac{1}{3}\end{pmatrix}$, $y=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$.

It follows from (1) that

$$Q(x)=x^TAx=(x_1\quad x_2\quad x_3)\begin{pmatrix}2&1&0\\1&1&-1\\0&-1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}, \qquad \textbf{(4)}$$

where $x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$, $A=\begin{pmatrix}2&1&0\\1&1&-1\\0&-1&1\end{pmatrix}$.

Rewrite (4) using (3)

$$Q(x)=x^TAx=(Uy)^TA(Uy)=y^TU^TAUy=y^T(U^TAU)y=y^TCy, \quad \textbf{(5)}$$

hence

$$C=U^TAU=\begin{pmatrix}1&0&0\\0&-\frac{1}{3}&\frac{2}{3}\\0&\frac{2}{3}&-\frac{1}{3}\end{pmatrix}^T\begin{pmatrix}2&1&0\\1&1&-1\\0&-1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-\frac{1}{3}&\frac{2}{3}\\0&\frac{2}{3}&-\frac{1}{3}\end{pmatrix}=$$

$$=\begin{pmatrix}1&0&0\\0&-\frac{1}{3}&\frac{2}{3}\\0&\frac{2}{3}&-\frac{1}{3}\end{pmatrix}\begin{pmatrix}2&1&0\\1&1&-1\\0&-1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-\frac{1}{3}&\frac{2}{3}\\0&\frac{2}{3}&-\frac{1}{3}\end{pmatrix}=\begin{pmatrix}2&1&0\\-\frac{1}{3}&-1&1\\\frac{2}{3}&1&-1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-\frac{1}{3}&\frac{2}{3}\\0&\frac{2}{3}&-\frac{1}{3}\end{pmatrix}=$$

$$=\begin{pmatrix}2&-1/3&2/3\\-1/3&1&-1\\2/3&-1&1\end{pmatrix} \qquad \textbf{(6)}$$

Substituting the expression for the matrix $C$ from (6) into (5)

$$Q(x)=y^TCy=(y_1\quad y_2\quad y_3)\begin{pmatrix}2&-\frac{1}{3}&\frac{2}{3}\\-\frac{1}{3}&1&-1\\\frac{2}{3}&-1&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=$$

$$=\left(2y_1-\frac{1}{3}y_2+\frac{2}{3}y_3\quad -\frac{1}{3}y_1+y_2-y_3\quad \frac{2}{3}y_1-y_2+y_3\right)\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=$$

$$=2y_1^2-\frac{1}{3}y_1y_2+\frac{2}{3}y_1y_3-\frac{1}{3}y_1y_2+y_2^2-y_2y_3+\frac{2}{3}y_1y_3-y_2y_3+y_3^2=$$

$$=2y_1^2-\frac{2}{3}y_1y_2+\frac{4}{3}y_1y_3-2y_2y_3+y_2^2+y_3^2=\tilde{Q}(y).$$

**Answer:** $\tilde{Q}(y)=2y_1^2-\frac{2}{3}y_1y_2+\frac{4}{3}y_1y_3-2y_2y_3+y_2^2+y_3^2$.